Find the Taylor Series for $f(x)$ centered at the given value of a. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.[1]

$$f(x) = \cos(x), \quad a = \frac{\pi}{2}$$

The general form for a Taylor series is

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \cdots \end{aligned}$$

Let's organize our work in a table. For this problem, we need enough entries in the table to recognize a pattern for $f^{(n)}\left(\frac{\pi}{2}\right)$ in terms of n.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{2}\right)$
0	$\cos(x)$	0
1	$-\sin(x)$	-1
2	$-\cos(x)$	0
3	$\sin(x)$	1
4	$\cos(x)$	0
5	$-\sin(x)$	-1
6	$-\cos(x)$	0
7	$\sin(x)$	1
8	$\cos(x)$	0
$\vdots$	$\vdots$	$\vdots$

Using the definition of a Taylor series and the values in the table, we get

$$\begin{aligned} f(x) &= 0 - \frac{1}{1!}\left(x-\frac{\pi}{2}\right)^1 + 0 + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 + 0 - \frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + 0 + \frac{1}{7!}\left(x-\frac{\pi}{2}\right)^7 + 0 + \cdots \\ &= -\frac{1}{1!}\left(x-\frac{\pi}{2}\right)^1 + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \frac{1}{7!}\left(x-\frac{\pi}{2}\right)^7 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{(2n+1)!}\left(x-\frac{\pi}{2}\right)^{2n+1} \end{aligned}$$

To find the radius and interval of convergence, we use the Ratio Test.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{\left(x-\frac{\pi}{2}\right)^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{\left(x-\frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}}\right|$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 771, #24.

$$
\begin{aligned}
&= \lim_{n\to\infty} \left| \frac{\left(x-\frac{\pi}{2}\right)^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{\left(x-\frac{\pi}{2}\right)^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n+1)!}{(2n+3)!} \cdot \frac{\left(x-\frac{\pi}{2}\right)^{2n+3}}{\left(x-\frac{\pi}{2}\right)^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{1}{(2n+3)\,(2n+2)} \cdot \left(x-\frac{\pi}{2}\right)^{2} \right| \\
&= \left(x-\frac{\pi}{2}\right)^{2} \cdot \lim_{n\to\infty} \left| \frac{1}{4n^2+10n+6} \right| \\
&= \left(x-\frac{\pi}{2}\right)^{2} \cdot 0 \\
&= 0
\end{aligned}
$$

The limit exists and is less than one for all values of x, and thus the series is convergent with $R = \infty$, and $IOC = (-\infty,\infty)$.

Just for fun, we've graphed the function $f(x) = \cos(x)$ in black and the 10th partial sum of our Taylor series in dotted red.