Evaluate the indefinite integral as a power series. What is the radius of convergence?[1]

$$\int \frac{t}{1+t^3}\ \mathrm{d}t$$

$$\begin{aligned}\frac{t}{1+t^3} &= t \cdot \frac{1}{1-(-t^3)} \\ &= t \cdot \sum_{n=0}^{\infty} \left(-t^3\right)^n \\ &= t \cdot \sum_{n=0}^{\infty} (-1)^n\, t^{3n} \\ &= \sum_{n=0}^{\infty} (-1)^n\, t^{3n+1}\end{aligned}$$

This series converges when $\left|-t^3\right| < 1$ or $|t| < 1$, so for this power series, $R = 1$ and the $IOC = (-1{,}1)$. (For this series, we do not need to check the endpoints because it was constructed from a geometric series.)

We integrate

$$\begin{aligned}\int \frac{t}{1+t^3}\ \mathrm{d}t &= \int \sum_{n=0}^{\infty} (-1)^n\, t^{3n+1}\ \mathrm{d}t \\ &= C + \sum_{n=0}^{\infty} (-1)^n\, \frac{t^{3n+2}}{3n+2}\end{aligned}$$

Since we integrated to find the power series, we know the radius of convergence R is the same as for the original power series, *i.e.* $R = 1$. However, the integration may change the convergence/divergence at the endpoints, so we check the new series at $t = -1$ and $t = 1$.

For $t = -1$,

$$\begin{aligned}\sum_{n=0}^{\infty} (-1)^n\, \frac{t^{3n+2}}{3n+2} &= \sum_{n=0}^{\infty} (-1)^n\, \frac{(-1)^{3n+2}}{3n+2} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^{4n+2}}{3n+2} \\ &= \sum_{n=0}^{\infty} \frac{1}{3n+2}\end{aligned}$$

We suspect divergence since the series is similar to the harmonic series. Let's try the Limit Comparison Test with the harmonic series.

$$\lim_{n\to\infty} \frac{\frac{1}{3n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{3n+2} = \lim_{n\to\infty} \frac{1}{3+\frac{2}{n}} = \frac{1}{3}$$

The limit is finite and positive. Since $\sum \frac{1}{n}$ is divergent, $\sum_{n=0}^{\infty} (-1)^n\, \frac{t^{3n+2}}{3n+2}$ is divergent when $t = -1$ by the Limit Comparison Test.

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[1] Stewart, *Calculus, Early Transcendentals*, p. 758, #26.

For $t = 1$,

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2} = \sum_{n=0}^{\infty} (-1)^n \frac{(1)^{3n+2}}{3n+2}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3n+2}$$

We suspect convergence since the series is similar to the alternating harmonic series. We could try the Ratio Test, but the limit will equal 1 (Go ahead, try it . . . you”ll see why.), so the Ratio Test will be inconclusive. The Alternating Series Test is our only hope (Well, that and Obi-Wan Kenobi.)

Here, $b_n = \frac{1}{3n+2}$ is positive, and if $f(x) = \frac{1}{3x+2}$, then

$$f'(x) = -\frac{1}{(3x+2)^2} \cdot 3$$
$$= -\frac{3}{(3x+2)^2}$$
$$< 0$$

so b_n is decreasing. Also

$$\lim_{n\to\infty} \frac{1}{3n+2} = 0$$

So the series $\sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2}$ is convergent when $t = 1$ by the Alternating Series Test.

Thus,

$$\int \frac{t}{1+t^3}\,\mathrm{d}t = C + \sum_{n=0}^{\infty} (-1)^n \frac{t^{3n+2}}{3n+2}, \quad t \in (-1{,}1]$$