Find the radius of convergence and the interval of convergence of the series.[1]

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\cdot 2^n}(x-1)^n$$

Let's apply the Ratio Test to our series.

$$\begin{aligned}
&\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\
&\quad = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{(2(n+1)-1)\cdot 2^{n+1}}(x-1)^{n+1}}{\frac{(-1)^n}{(2n-1)\cdot 2^n}(x-1)^n}\right| \\
&\quad = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(2(n+1)-1)\cdot 2^{n+1}}\cdot\frac{(2n-1)\cdot 2^n}{(-1)^n(x-1)^n}\right| \\
&\quad = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(-1)^n}\cdot\frac{(x-1)^{n+1}}{(x-1)^n}\cdot\frac{2n-1}{2n+1}\cdot\frac{2^n}{2^{n+1}}\right| \\
&\quad = \lim_{n\to\infty}\left|-1\cdot(x-1)\cdot\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\cdot\frac{1}{2}\right| \\
&\quad = 1\cdot|x-1|\cdot\frac{1}{2}\cdot\lim_{n\to\infty}\left|\frac{2-\frac{1}{n}}{2+\frac{1}{n}}\right| \\
&\quad = \frac{1}{2}|x-1|
\end{aligned}$$

The power series converges when

$$\begin{aligned}
\frac{1}{2}|x-1| &< 1 \\
|x-1| &< 2 \qquad (\text{So } R = 2.) \\
-2 < x-1 &< 2 \\
-1 < x &< 3
\end{aligned}$$

Now we test the power series for convergence at the endpoints.

When $x=-1$, our series becomes

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)\cdot 2^n}(-1-1)^n &= \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)\cdot 2^n}(-2)^n \\
&= \sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)\cdot 2^n}(-1)^n\cdot 2^n \\
&= \sum_{n=1}^{\infty}\frac{(-1\cdot -1)^n\cdot 2^n}{(2n-1)\cdot 2^n} \\
&= \sum_{n=1}^{\infty}\frac{1}{2n-1}
\end{aligned}$$

The terms of this series are positive and $\lim_{n\to\infty}\frac{1/(2n-1)}{1/n}=\frac{1}{2}<1$. So when $x=-1$, our series diverges by the Limit Comparison Test.

[1] Stewart, *Calculus, Early Transcendentals*, p. 751, #16.

When $x = 3$, our series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\cdot 2^n}(3-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\cdot 2^n}(2)^n$$
$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)}$$

This is an alternating series with $b_n = \frac{1}{2n-1}$. We have

$$2n+1 > 2n-1$$

so

$$\frac{1}{2n+1} < \frac{1}{2n-1}$$
$$\frac{1}{2(n+1)-1} < \frac{1}{2n-1}$$

so $b_{n+1} < b_n$, *i.e.*, the terms are decreasing. Also $\lim_{n\to\infty} \frac{1}{2n-1} = 0$. So when $x = 3$, our series is convergent by the Alternating Series Test.

Finally, the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)\cdot 2^n}(x-1)^n$$

has radius of convergence 2 and is convergent on the interval (−1,3].