Calculus II, Section 11.5, #26
Alternating Series

Show that the series is convergent. How many terms of the series do we need to add in order to find the sum to the indicated accuracy?[1]

$$\sum_{n=1}^{\infty}\left(-\frac{1}{n}\right)^n, \qquad |\text{error}| < 0.00005$$

We can write the series as

$$\sum_{n=1}^{\infty}(-1)^n\,\frac{1}{n^n}$$

so the series is alternating with $b_n = \frac{1}{n^n}$. Since $n \geq 1$, $b_n > 0$.

The corresponding function is

$$f(x) = \frac{1}{x^x} = x^{-x}$$

Using logarithmic differentiation,

$$f'(x) = -x^{-x}\left(1 + \ln(x)\right)$$

If

$$y = x^{-x}$$

then

$$\ln(y) = -x\ln(x)$$

so

$$\begin{aligned}\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} &= -x\cdot\frac{1}{x} + \ln(x)\cdot -1\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(-1-\ln(x)\right)\\ &= -x^{-x}\left(1+\ln(x)\right)\end{aligned}$$

so $f'(x) < 0$ and $\{b_n\}$ is decreasing. Finally,

$$\lim_{n\to\infty}\frac{1}{n^n} = 0$$

so our series is convergent by the Alternating Series Test.

Note that

$$b_5 = \frac{1}{5^5} = 0.00032$$

which is larger than the desired 0.00005.

$$b_6 = \frac{1}{6^6} \approx 0.00002143$$

which is smaller than the desired 0.00005, so we need to find s_5 to get the desired accuracy.

$$\begin{aligned}\sum_{n=1}^{\infty}(-1)^n\,\frac{1}{n^n} \approx s_5 &= \sum_{n=1}^{5}(-1)^n\,\frac{1}{n^n}\\ &= -\frac{1}{1^1}+\frac{1}{2^2}-\frac{1}{3^3}+\frac{1}{4^4}-\frac{1}{5^5}\\ &\approx -0.78345\end{aligned}$$

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[1] Stewart, *Calculus, Early Transcendentals*, p. 736, #26.