Test the series for convergence or divergence.[1]

$$\sum_{n=1}^{\infty} (-1)^{n+1}\, n\mathrm{e}^{-n}$$

We have

$$\sum_{n=1}^{\infty} (-1)^{n+1}\, n\mathrm{e}^{-n} = 1\mathrm{e}^{-1} - 2\mathrm{e}^{-2} + 3\mathrm{e}^{-3} - 4\mathrm{e}^{-4} + 5\mathrm{e}^{-5} - \cdots$$

Since some of the terms of this series are negative, none of the Integral Test, Comparison Test, or Limit Comparison Test are applicable. The terms do alternate between positive and negative, so let's try the Alternating Series Test.

For our series, we have $a_n = (-1)^{n+1}\, n\mathrm{e}^{-n}$. So $b_n = n\mathrm{e}^{-n} = \frac{n}{\mathrm{e}^n}$, and $b_n > 0$ for $n \geq 1$.

To show that $\{b_n\}$ is decreasing, *i.e.*, $b_{n+1} \leq b_n$, consider the corresponding function $f(x) = \frac{x}{\mathrm{e}^x}$. Then

$$\begin{aligned} f'(x) &= \frac{\mathrm{e}^x \cdot 1 - x \cdot \mathrm{e}^x}{(\mathrm{e}^x)^2} \\ &= \frac{\mathrm{e}^x (1 - x)}{\mathrm{e}^{2x}} \\ &= \frac{1 - x}{\mathrm{e}^x} \end{aligned}$$

Since $\mathrm{e}^x > 0$, $f'(x) < 0$ for $x > 1$. So the corresponding function is decreasing, and we know $\{b_n\}$ is decreasing.

Finally, we compute

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{\mathrm{e}^n}$$

This limit has the form $\frac{\infty}{\infty}$, so we apply l'Hospital's Rule to get

$$\begin{aligned} &= \lim_{n\to\infty} \frac{1}{\mathrm{e}^n} \\ &= 0 \end{aligned}$$

So the b_n are positive, decreasing, and have a limit of zero as n heads towards infinity. Thus, our series

$$\sum_{n=1}^{\infty} (-1)^{n+1}\, n\mathrm{e}^{-n}$$

is convergent by the Alternating Series Test.

[1] Stewart, *Calculus, Early Transcendentals*, p. 736, #12.