Use the Integral Test to determine whether the series is convergent or divergent.[1]

$$\sum_{n=1}^{\infty} n^2 e^{-n^3}$$

We need to show that the function $f(x)$ such that $a_n = f(n)$ is continuous, positive, and decreasing on the interval $[1,\infty)$.

For this specific series, $f(x) = x^2 e^{-x^3}$. This function is the product of continuous functions x^2 and e^{-x^3}, thus it is continuous. On $[1,\infty)$, the function is positive since it is the product of x^2 and an exponential function. Finally,

$$\begin{aligned} f'(x) &= x^2 \cdot e^{-x^3} \cdot -3x^2 + e^{-x^3} \cdot 2x \\ &= -3x^4 e^{-x^3} + 2x e^{-x^3} \\ &= x e^{-x^3} \left(-3x^3 + 2\right) \\ &= \frac{x\left(-3x^3+2\right)}{e^{x^3}} \\ &< 0 \end{aligned}$$

for $x \in [1,\infty)$. Thus the function is decreasing on $[1,\infty)$ and together with the other deductions, we can apply the Integral Test.

We evaluate

$$\int_1^{\infty} x^2 e^{-x^3}\, dx$$

Let $u = -x^3$, then $du = -3x^2\, dx$; when $x = 1$, $u = -1$ and when $x \to \infty$, $u \to -\infty$.

$$\begin{aligned} &= -\frac{1}{3} \int_{x=1}^{x\to\infty} -3x^2 e^{-x^3}\, dx \\ &= -\frac{1}{3} \int_{u=-1}^{u\to-\infty} e^u\, du \\ &= -\frac{1}{3} \lim_{t\to-\infty} \int_{u=-1}^{t} e^u\, du \\ &= -\frac{1}{3} \lim_{t\to-\infty} \Big[e^u\Big]_{u=-1}^{t} \\ &= -\frac{1}{3} \lim_{t\to-\infty} \left[e^t - e^{-1}\right] \\ &= -\frac{1}{3} \lim_{t\to-\infty} \left[e^t - \frac{1}{e}\right] \\ &= -\frac{1}{3} \left[0 - \frac{1}{e}\right] \\ &= \frac{1}{3e} \end{aligned}$$

So the improper integral is convergent.

Thus, by the Integral Test, the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ is convergent.

[1] Stewart, *Calculus, Early Transcendentals*, p. 725, #8.