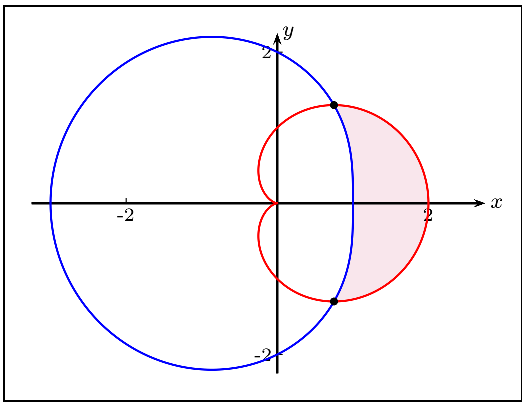
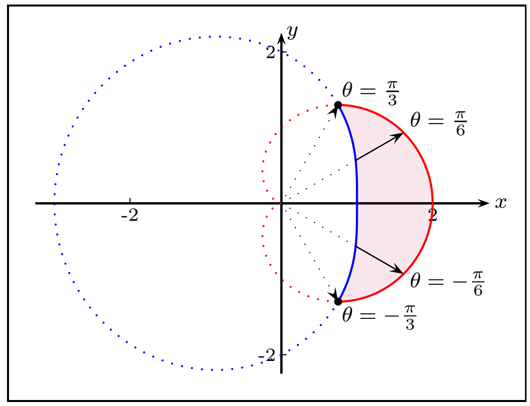
Find the area of the region that lies inside the first curve and outside the second curve.[1]

$$r = 1 + \cos(\theta), \quad r = 2 - \cos(\theta)$$





The region is shown in the first diagram. To find the points of intersection, we solve

$$\begin{aligned} 1 + \cos(\theta) &= 2 - \cos(\theta) \\ 2\cos(\theta) &= 1 \\ \cos(\theta) &= \frac{1}{2} \end{aligned}$$

From the unit circle, we get $\theta = \frac{\pi}{3}$ or $\theta = \frac{5\pi}{3}$, but those values of θ would sweep out the region starting in the first quadrant and moving throught the second and third quadrants to finish in the fourth quadrant... not the region we want! So we will integrate from $\theta = -\frac{\pi}{3}$ to $\theta = \frac{\pi}{3}$.

From the second diagram, we can see that the portion of the radius in which we are intertested has length $(1 + \cos(\theta)) - (2 - \cos(\theta))$. So the area A is given by

$$\begin{aligned} A &= \int_{\theta=-\pi/3}^{\theta=\pi/3} \frac{1}{2}\left[(1+\cos(\theta)) - (2-\cos(\theta))\right]^2 \, d\theta \\ &= \frac{1}{2}\int_{\theta=-\pi/3}^{\theta=\pi/3} (-1 + 2\cos(\theta))^2 \, d\theta \\ &= \frac{1}{2}\int_{\theta=-\pi/3}^{\theta=\pi/3} 1 - 4\cos(\theta) + 4\cos^2(\theta) \, d\theta \\ &= \frac{1}{2}\int_{\theta=-\pi/3}^{\theta=\pi/3} 1 - 4\cos(\theta) + 4\cdot\frac{1}{2}(1+\cos(2\theta)) \, d\theta \\ &= \frac{1}{2}\int_{\theta=-\pi/3}^{\theta=\pi/3} 1 - 4\cos(\theta) + 2 + 2\cos(2\theta) \, d\theta \\ &= \frac{1}{2}\int_{\theta=-\pi/3}^{\theta=\pi/3} 3 - 4\cos(\theta) + 2\cos(2\theta) \, d\theta \\ &= \frac{1}{2}\Big[3\theta - 4\sin(\theta) + \sin(2\theta)\Big]_{\theta=-\pi/3}^{\theta=\pi/3} \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 673, #26.

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\begin{aligned}
&= \frac{1}{2}\Big[\left(3\cdot\frac{\pi}{3} - 4\sin\left(\frac{\pi}{3}\right) + \sin\left(2\cdot\frac{\pi}{3}\right)\right) - \left(3\cdot -\frac{\pi}{3} - 4\sin\left(-\frac{\pi}{3}\right) + \sin\left(2\cdot -\frac{\pi}{3}\right)\right)\Big] \\
&= \frac{1}{2}\left[\pi - 2\sqrt{3} + \frac{\sqrt{3}}{2} + \pi - 2\sqrt{3} + \frac{\sqrt{3}}{2}\right] \\
&= \frac{1}{2}\left[2\pi - 3\sqrt{3}\right] \\
&= \frac{2\pi - 3\sqrt{3}}{2}
\end{aligned}
$$