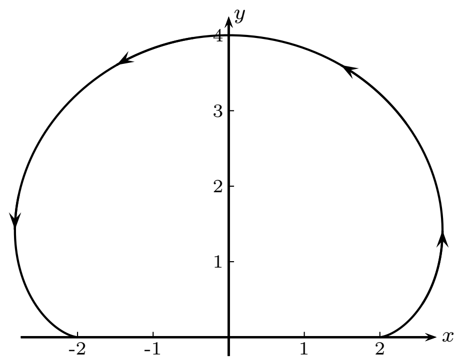
Find the exact length of the curve.[1]

$$x = 3\cos(t) - \cos(3t), \quad y = 3\sin(t) - \sin(3t), \quad 0 \le t \le \pi$$

The desired curve, traced out for $0 \le t \le \pi$, is shown at right. In terms of t, the arc length L is given by

$$L = \int_{t=0}^{t=\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$



We have

$$x = 3\cos(t) - \cos(3t)$$

so

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= -3\sin(t) + 3\sin(3t) \\
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 &= \left(-3\sin(t) + 3\sin(3t)\right)^2 \\
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 &= 9\sin^2(t) - 18\sin(t)\sin(3t) + 9\sin^2(3t)
\end{aligned}$$

and

$$y = 3\sin(t) - \sin(3t)$$

so

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= 3\cos(t) - 3\cos(3t) \\
\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= \left(3\cos(t) - 3\cos(3t)\right)^2 \\
\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 9\cos^2(t) - 18\cos(t)\cos(3t) + 9\cos^2(3t)
\end{aligned}$$

Adding

$$\begin{aligned}
\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 &= 9 - 18\left(\sin(t)\sin(3t) + \cos(t)\cos(3t)\right) + 9 \\
&= 18 - 18\cos(t - 3t) \\
&= 18\left(1 - \cos(-2t)\right) \\
&= 18\left(1 - \cos(2t)\right) \\
&= 18\left(1 - \left(1 - 2\sin^2(t)\right)\right) \\
&= 36\sin^2(t)
\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 656, #44.

which gives us

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2} = \sqrt{36\sin^2(t)}$$

and since $\sin(t) \geq 0$ on $0 \leq t \leq \pi$

$$= 6\sin(t)$$

Finally

$$\begin{aligned} L &= \int_{t=0}^{t=\pi} \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \\ &= [-6\cos(t)]_{t=0}^{t=\pi} \\ &= 6 - (-6) \\ &= 12 \end{aligned}$$