A tank with a capacity of $400\,\mathrm{L}$ *is full of a mixture or water and chlorine* (Cl) *with a concentration of* $0.05\,\mathrm{g}$ *of chlorine per liter. In order to reduce the concentration of chlorine, fresh water is pumped into the tank at the rate of* $4\,\mathrm{L/s}$*. The mixture is kept stirred and is pumped out at a rate of* $10\,\mathrm{L/s}$*. Find the amount of chlorine in the tank as a function of time.*[1]

Let $y(t) =$ amt. of Cl at time t.

At time $t = 0$, there is $(400\,\mathrm{L})\,(0.05\,\mathrm{g\!\!/L}) = 20\,\mathrm{g}$ of Cl in the tank, so the initial condition is $y(0) = 20$.

To set up our differential equation, we consider the rate at which Cl flows into the tank and the rate at which Cl flows out of the tank.

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= (\text{rate of Cl in}) - (\text{rate of Cl out})\\ &= \Big(0\,\mathrm{g/L}\Big)\Big(4\,\mathrm{L/s}\Big) - \left(\frac{y}{400-6t}\,\mathrm{gL}\right)\Big(10\,\mathrm{L/s}\Big)\end{aligned}$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= -\frac{10y}{400-6t}\\ \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{10y}{400-6t} &= 0\end{aligned}$$

The integrating factor is

$$\begin{aligned}\text{int. factor} &= \mathrm{e}^{\int \frac{10}{400-6t}\,\mathrm{d}t}\\ &= \mathrm{e}^{10\cdot -\frac{1}{6}\cdot\ln|400-6t|}\end{aligned}$$

Since $400 - 6t$ is the volume, we know $400 - 6t > 0$

$$\begin{aligned}&= \mathrm{e}^{10\cdot -\frac{1}{6}\cdot\ln(400-6t)}\\ &= \mathrm{e}^{-\frac{5}{3}\cdot\ln(400-6t)}\\ &= \mathrm{e}^{\ln(400-6t)^{-5/3}}\\ &= (400-6t)^{-5/3}\end{aligned}$$

Multiplying our FOLDE by this integrating factor, we get

$$\begin{aligned}(400-6t)^{-5/3}\frac{\mathrm{d}y}{\mathrm{d}t} + (400-6t)^{-5/3}\frac{10y}{400-6t} &= (400-6t)^{-5/3}\cdot 0\\ (400-6t)^{-5/3}\frac{\mathrm{d}y}{\mathrm{d}t} + (400-6t)^{-8/3}\cdot 10y &= 0\end{aligned}$$

So

$$\Big((400-6t)^{-5/3}\cdot y\Big)' = 0$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 626, #34.

and we integrate

$$\begin{aligned} \int \left((400 - 6t)^{-5/3} \cdot y \right)' \, \mathrm{d}t &= \int 0 \, \mathrm{d}t \\ (400 - 6t)^{-5/3} \cdot y &= C_1 \\ y &= C_1 \left(400 - 6t\right)^{5/3} \end{aligned}$$

From $y(0) = 20$, we know if $t = 0$, then $y = 20$. Substituting,

$$\begin{aligned} 20 &= C_1 \left(400 - 6 \cdot 0\right)^{5/3} \\ 20 &= C_1 \left(400\right)^{5/3} \\ C_1 &= \frac{20}{(400)^{5/3}} \end{aligned}$$

Substituting

$$\begin{aligned} y &= C_1 \left(400 - 6t\right)^{5/3} \\ y &= \frac{20}{(400)^{5/3}} \left(400 - 6t\right)^{5/3} \end{aligned}$$

Thus, the particular solution that gives the amount of Cl at time t is

$$y(t) = \frac{20}{(400)^{5/3}} \left(400 - 6t\right)^{5/3}$$