Solve the initial-value problem.[1]

$$xy' + y = x\ln(x), \quad y(1) = 0$$

We want to solve the first-order linear differential equation (FOLDE)

$$xy' + y = x\ln(x)$$

with the initial condition $y(1) = 0$. The function for which we wish to solve is y and the independent variable is x. Also, $\ln(x)$ imposes the condition $x > 0$ upon the independent variable.

We want to write the FOLDE in the form

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y &= Q(x)\\ xy' + y &= x\ln(x)\\ y' + \frac{1}{x}y &= \ln(x)\end{aligned}$$

The integrating factor is

$$\begin{aligned}\text{int. factor} &= \mathrm{e}^{\int \frac{1}{x}\,\mathrm{d}x}\\ &= \mathrm{e}^{\ln|x|}\end{aligned}$$

but since we know $x > 0$

$$\begin{aligned}&= \mathrm{e}^{\ln(x)}\\ &= x\end{aligned}$$

Multiplying our FOLDE by this integrating factor, we get

$$xy' + y = x\ln(x)$$

So

$$(xy)' = x\ln(x)$$

and we integrate

$$\begin{aligned}\int (xy)'\ \mathrm{d}x &= \int x\ln(x)\,\mathrm{d}x\\ xy &= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C_2\\ y &= \frac{1}{2}x\ln(x) - \frac{1}{4}x + \frac{C_2}{x}\end{aligned}$$

Integration by parts:

$$\int x\ln(x)\,\mathrm{d}x$$

Let $u = \ln(x)$, so $\mathrm{d}v = x\ \mathrm{d}x$. Then $\mathrm{d}u = \frac{1}{x}\ \mathrm{d}x$ and $v = \frac{x^2}{2}$.

$$\begin{aligned}&= \frac{1}{2}x^2\ln(x) - \int \frac{x^2}{2}\cdot\frac{1}{x}\,\mathrm{d}x\\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\int x\,\mathrm{d}x\\ &= \frac{1}{2}x^2\ln(x) - \left(\frac{1}{2}\cdot\frac{x^2}{2} + C_1\right)\\ &= \frac{1}{2}x^2\ln(x) - \frac{1}{4}x^2 + C_2\end{aligned}$$

From $y(1) = 0$, we know $x = 1$, then $y = 0$. Substituting,

$$\begin{aligned}0 &= \frac{1}{2}\cdot 1\cdot\ln(1) - \frac{1}{4}\cdot 1 + \frac{C_2}{1}\\ 0 &= -\frac{1}{4} + C_2\\ C_2 &= \frac{1}{4}\end{aligned}$$

Thus, the particular solution to $xy' + y = x\ln(x), \quad y(1) = 0$ is

$$y = \frac{1}{2}x\ln(x) - \frac{1}{4}x + \frac{1}{4x}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 625, #18.