Solve the differential equation.[1]

$$2xy' + y = 2\sqrt{x}$$

We want to solve the first-order linear differential equation (FOLDE)

$$2xy' + y = 2\sqrt{x}$$

The derivative in the equation is y', so the independent variable is x. This would be more clear if the author had written $y'(x)$ or $\frac{\mathrm{d}y}{\mathrm{d}x}$, but since there are no other parameters in the equation, the meaning is (mostly) clear.

Before we start, note that $\sqrt{x}$ imposes the condition $x \geq 0$ upon the independent variable x.

We want to write the FOLDE in the form

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} + P(x)\,y &= Q(x) \\ 2xy' + y &= 2\sqrt{x} \\ y' + \frac{1}{2x}y &= \frac{2\sqrt{x}}{2x} \\ y' + \frac{1}{2x}y &= \frac{1}{\sqrt{x}} \end{aligned}$$

Before we continue, note that $\frac{1}{\sqrt{x}}$ now imposes the condition $x > 0$ upon the independent variable x.

The integrating factor is

$$\begin{aligned} \text{int. factor} &= \mathrm{e}^{\int \frac{1}{2x}\,\mathrm{d}x} \\ &= \mathrm{e}^{\frac{1}{2}\int \frac{1}{x}\,\mathrm{d}x} \\ &= \mathrm{e}^{\frac{1}{2}\ln|x|} \end{aligned}$$

and since $x > 0$,

$$\begin{aligned} &= \mathrm{e}^{\frac{1}{2}\ln x} \\ &= \left(\mathrm{e}^{\ln x}\right)^{\frac{1}{2}} \\ &= x^{\frac{1}{2}} \\ &= \sqrt{x} \end{aligned}$$

Multiplying our FOLDE by this integrating factor, we get

$$\sqrt{x}\,y' + \frac{\sqrt{x}}{2x}y = \frac{\sqrt{x}}{\sqrt{x}}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 625, #10.

$$\sqrt{x}\,y' + \frac{1}{2\sqrt{x}}y = 1$$

So

$$\left(\sqrt{x}\,y\right)' = 1$$

and we integrate

$$\begin{aligned}\int \left(\sqrt{x}\,y\right)' \,\mathrm{d}x &= \int 1\mathrm{d}x \\ \sqrt{x}\,y &= x + C \\ y &= \frac{x + C}{\sqrt{x}}\end{aligned}$$

Thus the general solution to the FOLDE $2xy' + y = 2\sqrt{x}$ is $y = \frac{x+C}{\sqrt{x}}$.