Another model for a growth function for a limited population is given by the ***Gompertz function****, which is a solution of the differential equation*

$$\frac{\mathrm{d}P}{\mathrm{d}t} = c\ln\left(\frac{M}{P}\right)P$$

where c is a constant and M is the carrying capacity.[1]

(a) *Solve this differential equation.*

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}t} &= c\ln\left(\frac{M}{P}\right)P \\
\frac{1}{P\ln\left(\frac{M}{P}\right)}\,\mathrm{d}P &= c\,\mathrm{d}t \\
\int \frac{1}{P\ln\left(\frac{M}{P}\right)}\,\mathrm{d}P &= \int c\ \mathrm{d}t
\end{aligned}$$

Let $u = \ln\left(\frac{M}{P}\right)$, so $\mathrm{d}u = \frac{1}{\frac{M}{P}} \cdot -\frac{M}{P^2}\,\mathrm{d}P = \frac{P}{M} \cdot -\frac{M}{P^2}\,\mathrm{d}P = -\frac{1}{P}\,\mathrm{d}P$.

$$\begin{aligned}
-\int -\frac{1}{P\ln\left(\frac{M}{P}\right)}\,\mathrm{d}P &= \int c\ \mathrm{d}t \\
-\int \frac{1}{u}\,\mathrm{d}u &= \int c\ \mathrm{d}t \\
\int \frac{1}{u}\,\mathrm{d}u &= -\int c\ \mathrm{d}t \\
\ln|u| &= -\left[ct + C_1\right] \\
\ln|u| &= -ct + C_2 \\
|u| &= \mathrm{e}^{-ct+C_2} \\
|u| &= \mathrm{e}^{-ct}\cdot\mathrm{e}^{C_2} \\
|u| &= C_3\mathrm{e}^{-ct}
\end{aligned}$$

and because $|u| = C_3\mathrm{e}^{-ct} \implies u = \pm C_3\mathrm{e}^{-ct}$, we have

$$\begin{aligned}
u &= C_4\mathrm{e}^{-ct} \\
\ln\left(\frac{M}{P}\right) &= C_4\mathrm{e}^{-ct}
\end{aligned}$$

Now, at $t = 0$, $P = P_0$, and

$$\begin{aligned}
\ln\left(\frac{M}{P_0}\right) &= C_4\mathrm{e}^{-c\cdot 0} \\
\ln\left(\frac{M}{P_0}\right) &= C_4
\end{aligned}$$

So we have

$$\begin{aligned}
\ln\left(\frac{M}{P}\right) &= \ln\left(\frac{M}{P_0}\right)\mathrm{e}^{-ct} \\
\frac{M}{P} &= \mathrm{e}^{\ln\left(\frac{M}{P_0}\right)\mathrm{e}^{-ct}} \\
P &= M\mathrm{e}^{-\ln\left(\frac{M}{P_0}\right)\mathrm{e}^{-ct}}
\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 620, #22.

(b) *Compute* $\lim_{t\to\infty} P(t)$.

$$\begin{aligned}\lim_{t\to\infty} P(t) &= \lim_{t\to\infty} M\mathrm{e}^{-\ln\left(\frac{M}{P_0}\right)\mathrm{e}^{-ct}}\\ &= \lim_{t\to\infty} M\mathrm{e}^{-\ln\left(\frac{M}{P_0}\right)\cdot 0}\\ &= \lim_{t\to\infty} M\mathrm{e}^{0}\\ &= M\end{aligned}$$

(c) *Graph the Gompertz growth function for* $M = 1000$, $P_0 = 100$, *and* $c = 0.05$.