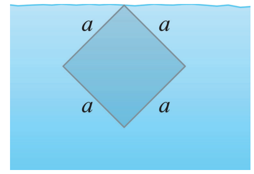
The vertical plate is submerged (or partially submerged) in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.[1]

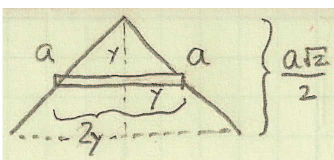


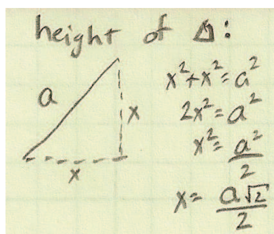
The square plate in the diagram is not given with any units. Rather than arbitrarily assigning units, we will express the weight density as

$$\text{density} \cdot \text{acceleration due to gravity} = \rho \cdot g = \delta$$

Also, the lines defining the edges of the plate change as the depth changes, so we will need to find the force on the top half of the plate with one integral and the force on the bottom half with another integral, and then add the two results together.

In Figure 1, we've drawn the representative rectangle for the top half of the plate.





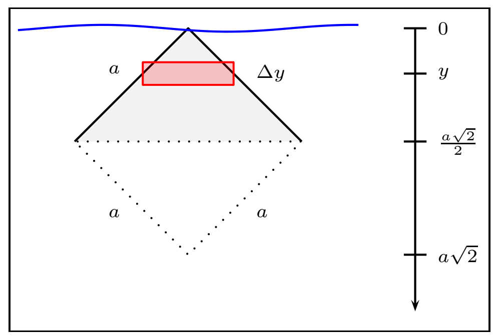


Figure 1

The top half of the plate is a 45-45-90 triangle, and from the geometry of that shape, we get the base of the representative rectangle to be $2y$. The height of the rep. rect. is Δy, so we have

$$\begin{aligned} \text{area of rep. rect.} &= \text{base of rep. rect.} \cdot \text{height of rep. rect.} \\ &= 2y \cdot \Delta y \end{aligned}$$

The force on the representative rectangle is given by

$$\begin{aligned} \text{force on rep. rect.} &= \text{weight density} \cdot \text{area} \cdot \text{depth} \\ &= \delta \cdot 2y \cdot \Delta y \cdot y \\ &= 2\delta \cdot y^2 \Delta y \end{aligned}$$

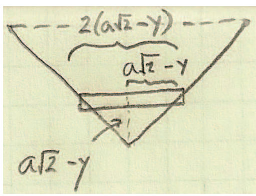
We create representative rectangles from $y = 0$ to $y = \frac{a\sqrt{2}}{2}$, so the total hydrostatic force, F, is given by

$$\begin{aligned} F &= \int_{y=0}^{y=\frac{a\sqrt{2}}{2}} 2\delta \cdot y^2 \, \mathrm{d}y \\ &= 2\delta \left[\frac{y^3}{3} \right]_{y=0}^{y=\frac{a\sqrt{2}}{2}} \\ &= 2\delta \left[\frac{\left(\frac{a\sqrt{2}}{2}\right)^3}{3} - \frac{(0)^3}{3} \right] \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 566, #10.

$$= 2\delta \left[\frac{\frac{a^3 \cdot 2\sqrt{2}}{8}}{3} - 0 \right]$$
$$= 2\delta \cdot \frac{a^3 \cdot 2\sqrt{2}}{24}$$
$$= \frac{a^3 \delta \sqrt{2}}{6}$$

In Figure 2, we've drawn the representative rectangle for the bottom half of the plate.



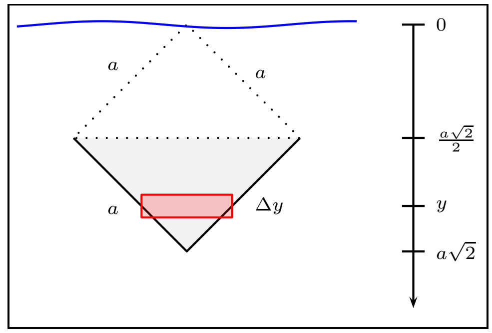


Figure 2

The bottom half of the plate is a 45-45-90 triangle, and from the geometry of that shape, we get the base of the representative rectangle to be $2\left(a\sqrt{2} - y\right)$. The height of the rep. rect. is Δy, so we have

$$\text{area of rep. rect.} = \text{base of rep. rect.} \cdot \text{height of rep. rect.}$$
$$= 2\left(a\sqrt{2} - y\right) \cdot \Delta y$$

The force on the representative rectangle is given by

$$\text{force on rep. rect.} = \text{weight density} \cdot \text{area} \cdot \text{depth}$$
$$= \delta \cdot 2\left(a\sqrt{2} - y\right) \cdot \Delta y \cdot y$$
$$= 2\delta \cdot \left(a\sqrt{2}y - y^2\right) \Delta y$$

We create representative rectangles from $y = \frac{a\sqrt{2}}{2}$ to $y = a\sqrt{2}$, so the total hydrostatic force, F, is given by

$$F = \int_{y=\frac{a\sqrt{2}}{2}}^{y=a\sqrt{2}} 2\delta \cdot \left(a\sqrt{2}y - y^2\right) \, \mathrm{d}y$$
$$= 2\delta \left[a\sqrt{2} \cdot \frac{y^2}{2} - \frac{y^3}{3} \right]_{y=\frac{a\sqrt{2}}{2}}^{y=a\sqrt{2}}$$
$$= 2\delta \left[\left(a\sqrt{2} \cdot \frac{\left(a\sqrt{2}\right)^2}{2} - \frac{\left(a\sqrt{2}\right)^3}{3} \right) - \left(a\sqrt{2} \cdot \frac{\left(\frac{a\sqrt{2}}{2}\right)^2}{2} - \frac{\left(\frac{a\sqrt{2}}{2}\right)^3}{3} \right) \right]$$
$$= 2\delta \left[a^3\sqrt{2} - \frac{2a^3\sqrt{2}}{3} - \frac{a^3\sqrt{2}}{4} + \frac{a^3\sqrt{2}}{12} \right]$$
$$= 2\delta \cdot \frac{2a^3\sqrt{2}}{12}$$
$$= \frac{a^3\delta\sqrt{2}}{3}$$

Thus, the total hydrostatic force on the plate is $\frac{a^3\delta\sqrt{2}}{6} + \frac{a^3\delta\sqrt{2}}{3} = \frac{a^3\delta\sqrt{2}}{2}$.