(a) *Find the arc length function for the curve* $y = \ln(\sin(x))$, $0 < x < \pi$, *with starting point* $\left(0, \frac{\pi}{2}\right)$.[1]

Since y is given as a function of x, we will use the arc length formula

$$L = \int_{x=a}^{x=b} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\, \mathrm{d}x$$

and the corresponding arc length function

$$s(x) = \int_{t=a}^{t=x} \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\, \mathrm{d}t$$

We have

$$y = \ln(\sin(x))$$

so

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sin(x)} \cdot \cos(x)\\ &= \cot(x)\end{aligned}$$

and

$$\begin{aligned}1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 &= 1 + (\cot(x))^2\\ &= \csc^2(x)\end{aligned}$$

so

$$\begin{aligned}\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} &= \sqrt{\csc^2(x)}\\ &= |\csc(x)|\end{aligned}$$

Since we are interested in the interval $0 < x < \pi$ and $\csc(x) \geq 0$ on that interval, we have

$$\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} = \csc(x)$$

Substituting into our arc length function,

$$\begin{aligned}s(x) &= \int_{t=\pi/2}^{t=x} \csc(t)\, \mathrm{d}t\\ &= \Big[\ln\left|\csc(t) - \cot(t)\right|\Big]_{t=\pi/2}^{t=x}\\ &= \ln\left|\csc(x) - \cot(x)\right| - \ln\left|\csc\left(\frac{\pi}{2}\right) - \cot\left(\frac{\pi}{2}\right)\right|\\ &= \ln\left|\csc(x) - \cot(x)\right| - \ln\left|1 - 0\right|\\ s(x) &= \ln\left|\csc(x) - \cot(x)\right|\end{aligned}$$

and since $\csc(x) \geq \cot(x)$ for $0 < x < \pi$, we have

> We know $\sin(x) \leq \sin(x)$ so $\frac{1}{\sin(x)} \geq \frac{1}{\sin(x)}$. Also, $1 \geq \cos(x)$ so $\frac{1}{\sin(x)} \geq \frac{\cos(x)}{\sin(x)}$. Thus, $\csc(x) \geq \cot(x)$.

$$s(x) = \ln \csc(x) - \cot(x)$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 549, #36.

(b) *Graph both the curve and its arc length function on the same screen.*

Using WolframAlpha,