Make a substitution to express the integrand as a rational function and then evaluate the integral.[1]

$$\int \frac{\sin(x)}{\cos^2(x) - 3\cos(x)}\,\mathrm{d}x$$

Is the integrand one of our basic indefinite integrals? No. How about a basic u-substitution? Maybe. Integration by parts? No. Powers of trig functions? No. Does the integrand include a trig. sub. radical? No. Is the integrand a rational function? No.

Let's try a u-substitution. Let $u = \cos(x)$, so $\mathrm{d}u = -\sin(x)$.

$$\begin{aligned}
\int \frac{\sin(x)}{\cos^2(x) - 3\cos(x)}\,\mathrm{d}x &= -1 \cdot \int \frac{-\sin(x)}{\cos^2(x) - 3\cos(x)}\,\mathrm{d}x \\
&= -1 \cdot \int \frac{1}{u^2 - 3u}\,\mathrm{d}u \\
&= -1 \cdot \int \frac{1}{u(u-3)}\,\mathrm{d}u
\end{aligned}$$

Now we'll determine the partial fraction decomposition for the integrand. u is a distinct linear factor, as is $u - 3$, so we get

$$\frac{1}{u(u-3)} = \frac{A}{u} + \frac{B}{u-3}$$

The LCD is $u(u-3)$, and we multiply both sides of this identity by the LCD to get

$$\begin{aligned}
\frac{1}{u(u-3)} \cdot u(u-3) &= \frac{A}{u} \cdot u(u-3) + \frac{B}{u-3} \cdot u(u-3) \\
1 &= A(u-3) + Bu \\
1 &= Au - 3A + Bu \\
1 &= (A+B)u - 3A
\end{aligned}$$

Since this equation is an identity, *i.e.*, it is true for all allowable values of x, we equate coefficients to get the system of equations

$$\begin{cases} 0 = A + B \\ 1 = -3A \end{cases}$$

From the second equation we get $A = -\frac{1}{3}$. From the first equation, we get $B = \frac{1}{3}$.

The partial fraction decomposition is

$$\frac{1}{u(u-3)} = \frac{-1/3}{u} + \frac{1/3}{u-3}$$

so our integral becomes

$$\begin{aligned}
-1 \cdot \int \frac{1}{u(u-3)}\,\mathrm{d}u &= -1 \cdot \int \frac{-1/3}{u} + \frac{1/3}{u-3}\,\mathrm{d}u \\
&= -1 \cdot \left(-\frac{1}{3} \int \frac{1}{u}\,\mathrm{d}u + \frac{1}{3} \int \frac{1}{u-3}\,\mathrm{d}u \right)
\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 501, #48.

$$= -1 \cdot \left(-\frac{1}{3} \ln |u| + \frac{1}{3} \ln |u-3| + C \right)$$
$$= \frac{1}{3} \ln |u| - \frac{1}{3} \ln |u-3| + C$$

and since $u = \cos(x)$

$$= \frac{1}{3} \ln |\cos(x)| - \frac{1}{3} \ln |\cos(x) - 3| + C$$

Thus,

$$\int \frac{\sin(x)}{\cos^2(x) - 3\cos(x)} \, \mathrm{d}x = \frac{1}{3} \ln |\cos(x)| - \frac{1}{3} \ln |\cos(x) - 3| + C$$