Evaluate the integral.[1]

$$\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x$$

Is the integrand one of our basic indefinite integrals? No. How about a basic u-substitution? No. Integration by parts? No. Powers of trig functions? No. Does the integrand include a trig. sub. radical? No. Is the integrand a rational function? YES!

The degree of the numerator is 3, and the degree of the denominator is 4. Since the degree of the numerator is less than the degree of the denominator, we are ready to begin the partial fractions process.

Consider the integrand

$$\frac{x^3+6x-2}{x^4+6x^2} = \frac{x^3+6x-2}{x^2\left(x^2+6\right)}$$

x^2 is a repeated linear factor, whereas x^2+6 is a distinct, irreducible quadratic factor. We have

$$\frac{x^3+6x-2}{x^2\left(x^2+6\right)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+6}$$

The LCD is $x^2\left(x^2+6\right)$, and we multiply both sides of this identity by the LCD to get

$$\begin{aligned}
\frac{x^3+6x-2}{x^2\left(x^2+6\right)} \cdot x^2\left(x^2+6\right) &= \frac{A}{x}\cdot x^2\left(x^2+6\right) + \frac{B}{x^2}\cdot x^2\left(x^2+6\right) + \frac{Cx+D}{x^2+6}\cdot x^2\left(x^2+6\right) \\
x^3+6x-2 &= Ax\left(x^2+6\right) + B\left(x^2+6\right) + (Cx+D)\,x^2 \\
x^3+6x-2 &= A\left(x^3+6x\right) + B\left(x^2+6\right) + (Cx+D)\,x^2 \\
x^3+6x-2 &= Ax^3+6Ax+Bx^2+6B+Cx^3+Dx^2 \\
x^3+6x-2 &= Ax^3+Cx^3+Bx^2+Dx^2+6Ax+6B \\
x^3+6x-2 &= (A+C)\,x^3+(B+D)\,x^2+6Ax+6B
\end{aligned}$$

Since this equation is an identity, *i.e.*, it is true for all allowable values of x, we equate coefficients to get the system of equations

$$\left\{\begin{array}{rcl}
1 &=& A + C \\
0 &=& B + D \\
6 &=& 6A \\
-2 &=& 6B
\end{array}\right.$$

From the third equation we get $A=1$. From the fourth we get $B=-\frac{1}{3}$. Substituting our value for A into the first equation gives us $C=0$. Substituting our value for B into the second equation gives us $D=\frac{1}{3}$.

The partial fraction decomposition is

$$\begin{aligned}
\frac{x^3+6x-2}{x^2\left(x^2+6\right)} &= \frac{1}{x} + \frac{-1/3}{x^2} + \frac{0\cdot x+1/3}{x^2+6} \\
&= \frac{1}{x} + \frac{-1/3}{x^2} + \frac{1/3}{x^2+6}
\end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 501, #28.

so our integral becomes

$$\begin{aligned}\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x &= \int \frac{1}{x} + \frac{-1/3}{x^2} + \frac{1/3}{x^2+6}\,\mathrm{d}x \\ &= \int \frac{1}{x}\,\mathrm{d}x + \int \frac{-1/3}{x^2}\,\mathrm{d}x + \int \frac{1/3}{x^2+6}\,\mathrm{d}x \\ &= \int \frac{1}{x}\,\mathrm{d}x - \frac{1}{3}\int \frac{1}{x^2}\,\mathrm{d}x + \frac{1}{3}\int \frac{1}{x^2+6}\,\mathrm{d}x \\ &= \int \frac{1}{x}\,\mathrm{d}x - \frac{1}{3}\int x^{-2}\,\mathrm{d}x + \frac{1}{3}\int \frac{1}{x^2+\left(\sqrt{6}\right)^2}\,\mathrm{d}x\end{aligned}$$

The first integral is a straightforward natural logarithm, the second integral is a straightforward power rule, and the third integral is an inverse tangent form.

$$\begin{aligned}&= \ln|x| - \frac{1}{3}\left[-\frac{1}{x}\right] + \frac{1}{3}\cdot\frac{1}{\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + C \\ &= \ln|x| + \frac{1}{3x} + \frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + C\end{aligned}$$

Thus,

$$\int \frac{x^3+6x-2}{x^4+6x^2}\,\mathrm{d}x = \ln|x| + \frac{1}{3x} + \frac{1}{3\sqrt{6}}\tan^{-1}\left(\frac{x}{\sqrt{6}}\right) + C$$