Evaluate the integral.[1]

$$\int_0^{\pi/2} \frac{\cos(t)}{\sqrt{1+\sin^2(t)}}\,\mathrm{d}t$$

Does the integrand match one of our basic indefinite integral patterns? No. Can we do a basic u-substitution? Yes!

$$\int_0^{\pi/2} \frac{\cos(t)}{\sqrt{1+\sin^2(t)}}\,\mathrm{d}t$$

Let $u = \sin(t)$, so $\mathrm{d}u = \cos(t)\,\mathrm{d}t$, and when $t = 0$, $u = 0$, and $t = \frac{\pi}{2}$, $u = 1$. We get

$$\int_{u=0}^{u=1} \frac{1}{\sqrt{1+u^2}}\,\mathrm{d}u$$

For this integral, does the integrand match one of our basic indefinite integral patterns? No. Can we do a basic u-substitution? No. How about integration by parts? No. Is the integrand powers of trig functions? Nope. Does the integrand include a square root that matches one of our three trig substitutions? Yes!

The radical has the form $\sqrt{a^2+x^2}$, so we let $u = \tan(\theta)$ and $\mathrm{d}u = \sec^2(\theta)\,\mathrm{d}\theta$. Also, when $u = 0$, $0 = \tan(\theta)$ so $\theta = 0$ and when $u = 1$, $1 = \tan(\theta)$ so $\theta = \frac{\pi}{4}$. (Here, we chose the solutions to $0 = \tan(\theta)$ and $1 = \tan(\theta)$ that are in Quadrant I because that will fit the restrictions that come from simplifying the square root.) We get

$$\begin{aligned}
\sqrt{1+u^2} &= \sqrt{1+\tan^2(\theta)} \\
&= \sqrt{\sec^2(\theta)} \\
&= |\sec(\theta)| \\
&= \sec(\theta), \quad \theta \in \text{QI, IV}
\end{aligned}$$

So our integral becomes

$$\begin{aligned}
&\int_{\theta=0}^{\theta=\pi/4} \frac{1}{\sec(\theta)} \sec^2(\theta)\;\mathrm{d}\theta \\
&= \int_{\theta=0}^{\theta=\pi/4} \sec(\theta)\;\mathrm{d}\theta \\
&= \left[\ln|\sec(\theta)+\tan(\theta)|\right]_{\theta=0}^{\theta=\pi/4} \\
&= \ln\left|\sec\left(\frac{\pi}{4}\right)+\tan\left(\frac{\pi}{4}\right)\right| - \ln|\sec(0)+\tan(0)| \\
&= \ln\left|\sqrt{2}+1\right| - \ln|1+0| \\
&= \ln\left|\sqrt{2}+1\right|
\end{aligned}$$

Thus,

$$\int_0^{\pi/2} \frac{\cos(t)}{\sqrt{1+\sin^2(t)}}\,\mathrm{d}t = \ln\left(\sqrt{2}+1\right)$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 491, #30.