Evaluate the integral.[1]

$$\int (\arcsin(x))^2 \, \mathrm{d}x$$

$$I = \int (\arcsin(x))^2 \, \mathrm{d}x$$

Let $u = (\arcsin(x))^2$ and $\mathrm{d}v = \mathrm{d}x$. Then $\mathrm{d}u = 2\arcsin(x) \cdot \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$ and $v = x$. We have

$$\begin{aligned} I &= (\arcsin(x))^2 \cdot x - \int x \cdot 2\arcsin(x) \cdot \frac{1}{\sqrt{1-x^2}}\mathrm{d}x \\ &= x(\arcsin(x))^2 - \int \frac{2x \cdot \arcsin(x)}{\sqrt{1-x^2}}\mathrm{d}x \\ &= x(\arcsin(x))^2 - 2\int \frac{x \cdot \arcsin(x)}{\sqrt{1-x^2}}\mathrm{d}x \qquad (1) \end{aligned}$$

Now let's work on this last integral.

$$I_1 = \int \frac{x \cdot \arcsin(x)}{\sqrt{1-x^2}}\mathrm{d}x \qquad (2)$$

Let $t = \arcsin(x)$ (so $x = \sin(t)$), and $\mathrm{d}t = \frac{1}{\sqrt{1-x^2}}\mathrm{d}x$, and we have

$$\int \frac{x \cdot \arcsin(x)}{\sqrt{1-x^2}}\mathrm{d}x = \int t\sin(t) \, \mathrm{d}t$$

This latest integral looks like a perfect candidate for integration by parts.

$$I_2 = \int t\sin(t) \, \mathrm{d}t$$

Let $U = t$, so $\mathrm{d}v = \sin(t)\,\mathrm{d}t$ and $\mathrm{d}U = \mathrm{d}t$ and $V = -\cos t$. Thus

$$\begin{aligned} I_2 &= t \cdot -\cos t - \int -\cos t \, \mathrm{d}t \\ &= -t\cos t + \int \cos t \, \mathrm{d}t \\ &= -t\cos t + \sin(t) + C \qquad (3) \end{aligned}$$

Now we need to get this result back in terms of x. We can use $x = \sin(t)$, and draw a right triangle to find $\cos(t)$. From the figure, we see $\cos(t) = \frac{\sqrt{1-x^2}}{1}$. Substituting this back into (3) gives us

$$\begin{aligned} I_2 &= -t\cos t + \sin(t) + C \\ &= -\arcsin(x) \cdot \sqrt{1-x^2} + x + C \\ &= -\arcsin(x)\sqrt{1-x^2} + x + C \qquad (4) \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 476, #22.

Substituting (4) back into (2) gives us

$$\begin{aligned} I_1 &= \int \frac{x \cdot \arcsin(x)}{\sqrt{1-x^2}} \mathrm{d}x \\ &= -\arcsin(x)\sqrt{1-x^2} + x + C \end{aligned}$$

and substituting into (1), we have

$$\begin{aligned} &= x\left(\arcsin(x)\right)^2 - 2\left(-\arcsin(x)\sqrt{1-x^2} + x + C\right) \\ &= x\left(\arcsin(x)\right)^2 + 2\arcsin(x)\sqrt{1-x^2} - 2x + C \end{aligned}$$

where $-2 \cdot$ a constant $C =$ a constant, which we will just call C.

Thus,

$$\int \left(\arcsin(x)\right)^2 \ \mathrm{d}x = x\left(\arcsin(x)\right)^2 + 2\arcsin(x)\sqrt{1-x^2} - 2x + C$$