The region bounded by the given curves is rotated about the specified axis. Find the volume of the resulting solid by any method.[1]

$$y^2 - x^2 = 1, \quad y = 2; \quad \textit{about the } x\textit{-axis.}$$

The curve $y^2 - x^2 = 1$ is a hyperbola that opens in the y-direction. In Figure 1, the hyperbola and the line $y = 2$ are sketched along with the axis of revolution. From the figure, the lower branch of the hyperbola is not a boundary of the region. We solve the hyperbola equation for y:

$$\begin{aligned} y^2 - x^2 &= 1 \\ y^2 &= 1 + x^2 \end{aligned}$$

so

$$y = -\sqrt{1+x^2} \quad \text{or} \quad y = \sqrt{1+x^2}$$

The region bounded by the curves is in Quadrants I and II, so the values of y are all positive. Thus,

$$y = \sqrt{1+x^2}$$

Now we can find the points of intersection by solving

$$\begin{aligned} \sqrt{1+x^2} &= 2 \\ 1 + x^2 &= 4 \\ x^2 &= 3 \end{aligned}$$

so

$$x = -\sqrt{3} \text{ or } x = \sqrt{3}$$

Thus the points of intersection are $\left(-\sqrt{3}, 2\right)$ and $\left(\sqrt{3}, 2\right)$.

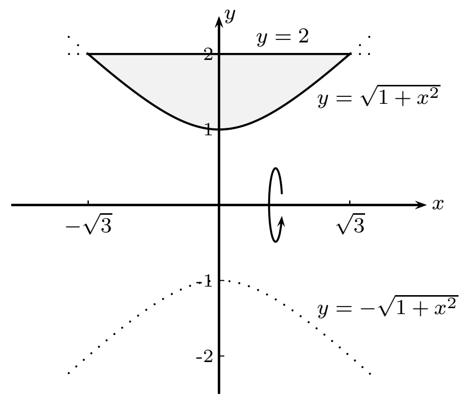

Figure 1: *The curves and the region.*

For this problem, we are not told which of washers or shells we should use. (As we would encounter in any real world problem.) Let's set up the integral for both methods, and then we'll decide which to compute.

In Figure 2, we have drawn a representative rectangle perpendicular to the axis of revolution. This will generate a washer of height Δx. The outside radius of the washer is $r_{\text{out}} = 2 - 0 = 2$, while the inside radius is $r_{\text{in}} = \sqrt{1+x^2} - 0 = \sqrt{1+x^2}$. We have

$$\begin{aligned} V_{\text{washer}} &= \left(\pi\,(2)^2 - \pi\left(\sqrt{1+x^2}\right)^2\right)\Delta x \\ &= \left(4\pi - \pi\left(1+x^2\right)\right)\Delta x \end{aligned}$$

and we generate washers from $x = -\sqrt{3}$ to $x = \sqrt{3}$, so the volume of the solid of revolution, V, is given by

$$\begin{aligned} V &= \int_{x=-\sqrt{3}}^{x=\sqrt{3}} 4\pi - \pi\left(1+x^2\right)\,\mathrm{d}x \\ &= \pi\int_{x=-\sqrt{3}}^{x=\sqrt{3}} 3 - x^2\,\mathrm{d}x \end{aligned}$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 455, #39.

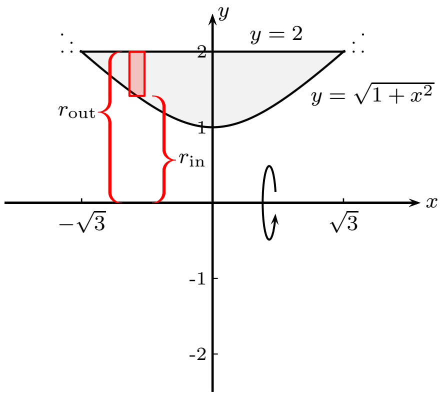

Figure 2: *The region and a representative rectangle for a washer.*

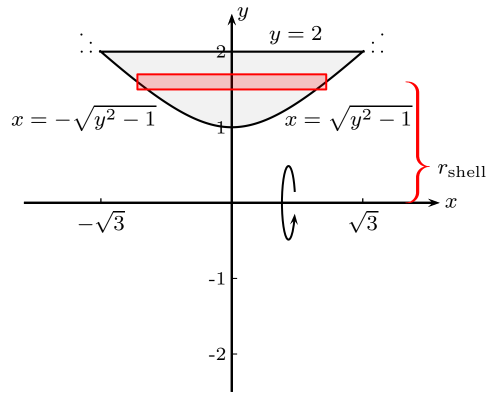

Figure 3: *The region and a representative rectangle for a shell.*

In Figure 3, we've drawn a representive rectangle parallel to the axis of revolution. This will generate a shell of thickness Δy. The volume of the shell is given by

$$V_{\text{shell}} = (2\pi y)\left(\sqrt{y^2-1} - \left(-\sqrt{y^2-1}\right)\right)(\Delta y)$$

We create shells from $y = 1$ to $y = 2$, so the volume, V, of the solid of revolution is given by

$$\begin{aligned} V &= \int_{y=1}^{y=2} (2\pi y)\left(\sqrt{y^2-1} - \left(-\sqrt{y^2-1}\right)\right)\,\mathrm{d}y \\ &= 2\pi \int_{y=1}^{y=2} y \cdot 2\sqrt{y^2-1}\,\mathrm{d}y \\ &= 4\pi \int_{y=1}^{y=2} y\sqrt{y^2-1}\,\mathrm{d}y \end{aligned}$$

So now we have the two integrals

$$V = \pi \int_{x=-\sqrt{3}}^{x=\sqrt{3}} 3 - x^2\,\mathrm{d}x \quad \text{or} \quad V = 4\pi \int_{y=1}^{y=2} y\sqrt{y^2-1}\,\mathrm{d}y$$

The left (washer) integral is a basic antiderivative and fundamental theorem of calculus, while the right (shell) integral requires a substitution to complete.

Let's compute

$$\begin{aligned} V &= \pi \int_{x=-\sqrt{3}}^{x=\sqrt{3}} 3 - x^2\,\mathrm{d}x \\ &= \pi \left[3x - \frac{x^3}{3}\right]_{x=-\sqrt{3}}^{x=\sqrt{3}} \\ &= \pi \left[\left(3\sqrt{3} - \frac{\left(\sqrt{3}\right)^3}{3}\right) - \left(3\cdot -\sqrt{3} - \frac{\left(-\sqrt{3}\right)^3}{3}\right)\right] \\ &= \pi \left[3\sqrt{3} - \frac{3\sqrt{3}}{3} + 3\sqrt{3} - \frac{3\sqrt{3}}{3}\right] \\ &= \pi \left[4\sqrt{3}\right] \\ &= 4\pi\sqrt{3} \end{aligned}$$

Now let's compute

$$V = 4\pi \int_{y=1}^{y=2} y\sqrt{y^2 - 1}\,\mathrm{d}y$$

Since we don't know a basic antiderivative for the integrand $y\sqrt{y^2-1}$, we'll try substitution. Let $u = y^2 - 1$, so $\mathrm{d}u = 2y\,\mathrm{d}y$.

$$\begin{aligned} V &= 4\pi \cdot \frac{1}{2} \int_{y=1}^{y=2} 2 \cdot y\sqrt{y^2 - 1}\,\mathrm{d}y \\ &= 2\pi \int_{y=1}^{y=2} \sqrt{y^2 - 1} \cdot 2y\,\mathrm{d}y \end{aligned}$$

When $y = 1$, $u = (1)^2 - 1 = 0$, and when $y = 2$, $u = (2)^2 - 1 = 3$. We have

$$\begin{aligned} V &= 2\pi \int_{u=0}^{u=3} \sqrt{u}\,\mathrm{d}u \\ &= 2\pi \int_{u=0}^{u=3} u^{1/2}\,\mathrm{d}u \\ &= 2\pi \left[\frac{u^{3/2}}{3/2}\right]_{u=0}^{u=3} \\ &= 2\pi \left[\frac{2u^{3/2}}{3}\right]_{u=0}^{u=3} \\ &= 2\pi \left[\frac{2 \cdot 3^{3/2}}{3} - \frac{2 \cdot 0^{3/2}}{3}\right] \\ &= 2\pi \left[2\sqrt{3} - 0\right] \\ &= 4\pi\sqrt{3} \end{aligned}$$

Thus, the volume of the solid formed when the region bounded by the curves $y^2 - x^2 = 1$ and $y = 2$ is revolved about the x-axis is $4\pi\sqrt{3}\,\text{units}^3$.

Students often ask "So which method is better?" or "Which method is easier?" The answer to these questions is "Neither." For this problem, both methods gave us integrals that we were able to compute and neither of the integrals was significantly more challenging.