Use the method of cylindrical shells to find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.[1]

$$x = -3y^2 + 12y - 9, \quad x = 0$$

The region to be rotated is bounded by the y-axis ($x = 0$) and a parabola with horizontal axis of symmetry that opens to the left. The curves[2] are sketched[3] in Figure 1.

Since we are going to use cylindrical shells, the representative rectangle is parallel to the axis of revolution; this is sketched in Figure 2. The width of the representative rectangle is a value of y, so everything in our integral will have to be in terms of y. Fortunately, our functions already gives us x as a function of y, so no additional algebra is necessary.

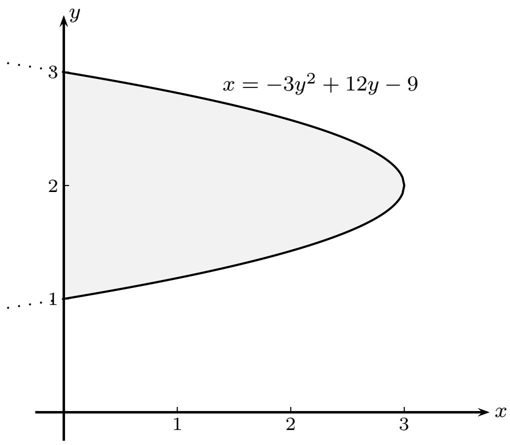

Figure 1: *The curves and the region.*

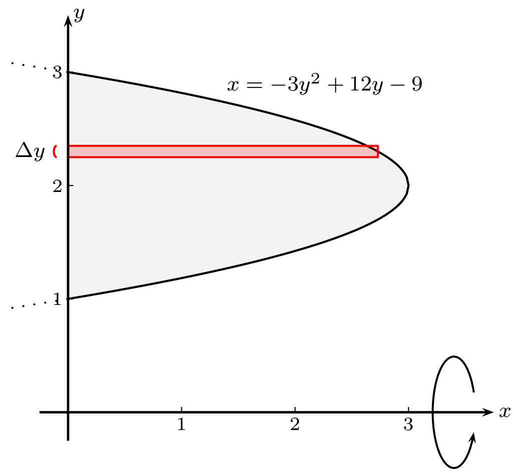

Figure 2: *The region with a representative rectangle.*

The problem directs us to use the method of cylindrical shells. This is a very good choice. We could do this problem using washers with representative rectangles perpendicular to the axis of revolution, but we would have to solve $x = -3y^2 + 12y - 9$ for y as a function of x (not very nice[4]) and then integrate a very messy function. Not impossible, but not at all user friendly.

In Figure 3, we've sketched the shell generated by revolving the representative rectangle about the x-axis. To the right is sketch the shell after is has been cut and flattened into a (very skinny) box. the volume of the shell is given by

$$V_{\text{shell}} = (\text{length of box})\,(\text{width of box})\,(\text{thickness of box})$$

and since the length of the box is the circumference of the shell

$$= (2\pi y)\left(-3y^2 + 12y - 9\right)(\Delta y)$$

We create shells from $y = 1$ to $y = 3$, so the volume of the solid of revolution is given by

$$V = \int_{y=1}^{y=3} 2\pi y \left(-3y^2 + 12y - 9\right)\, \mathrm{d}y$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 454, #12.

[2] In higher mathematics, the word "curve" is used in a more general sense to refer to any graph, curvy or not. Even though $x = 0$ is a straight line, we refer to it as a "curve".

[3] To sketch the parabola, we solved $0 = -3y^2 + 12y - 9$ to find the y-intercepts, and then used $y = -\frac{b}{2a} = -\frac{12}{-6} = 2$ to find the vertex at (3,2).

[4] We get $y = \frac{-12-\sqrt{144-12(9+x)}}{-6}$ or $y = \frac{-12+\sqrt{144-12(9+x)}}{-6}$.

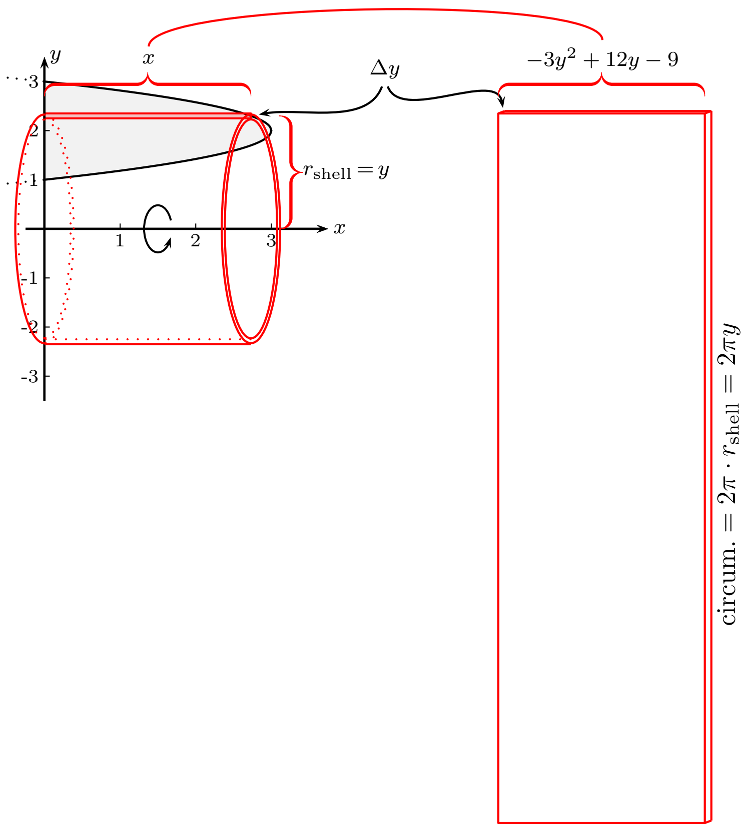

Figure 3: *The shell corresponding to the representative rectangle.*

$$
\begin{aligned}
V &= 2\pi \int_{y=1}^{y=3} -3y^3 + 12y^2 - 9y \, \mathrm{d}y \\
&= 2\pi \left[-3 \cdot \frac{y^4}{4} + 12 \cdot \frac{y^3}{3} - 9 \cdot \frac{y^2}{2} \right]_{y=1}^{y=3} \\
&= 2\pi \left[\frac{-3y^4}{4} + 4y^3 - \frac{9y^2}{2} \right]_{y=1}^{y=3} \\
&= 2\pi \left[\left(\frac{-3\,(3)^4}{4} + 4\,(3)^3 - \frac{9\,(3)^2}{2} \right) - \left(\frac{-3\,(1)^4}{4} + 4\,(1)^3 - \frac{9\,(1)^2}{2} \right) \right] \\
&= 2\pi \left[\left(\frac{-243}{4} + 108 - \frac{81}{2} \right) - \left(\frac{-3}{4} + 4 - \frac{9}{2} \right) \right] \\
&= 2\pi \left[\frac{-243}{4} + \frac{432}{4} - \frac{162}{4} + \frac{3}{4} - \frac{16}{4} + \frac{18}{4} \right] \\
&= 2\pi \cdot \frac{32}{4} \\
&= 16\pi
\end{aligned}
$$

Thus the volume of the solid obtained by rotating the region bounded by $x = -3y^2 + 12y - 9$ and $x = 0$ about the x-axis is 16π units^3.