Find the volume of the solid whose base S is the triangular region with vertices (0,0), (1,0), and (0,1). Cross-sections perpendicular to the y-axis are equilateral triangles.[1]

The described region is *not* a volume created by revolution about an axis, but the concept is the same. We will

1. Find the volume of a representative cross-section.
2. Determine the limits for which the cross-sections are generated.
3. Find the limit of the sum of the volumes as the number of cross-sections increases without bound, *i.e.*, write and compute the definite integral.

To begin finding the volume of a representative cross-section, let's sketch the base S of the solid. See Figure 1. The equation of the line is $x + y = 1$.

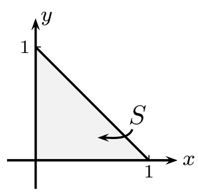

Figure 1: *Base S of the region.*

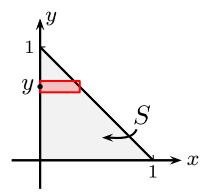

Figure 2: *Rep. rect. in the base S.*

In Figure 2, we've sketched a representative rectangle perpendicular to the y-axis. This will be the base of our representative cross-section. Since this representative rectangle is perpendicular to the y-axis, we need to express x as a function of y. We solve $x+y = 1$ for x and get $x = -y+1$. So the length of the representative rectangle is $-y + 1$ and the width is Δy.

Figure 3 shows the base S and the representative rectangle drawn in three dimensions. This is the standard orientation for three coordinate axes. You'll see axes drawn as in Figure 3 in further math classes, physics, engineering, and any other class that uses three dimensions to display information.

We are told that the cross section on the representative rectangle is an equilateral triangle. Figure 4 shows the cross-section built on top of the representative rectangle. We need to find an expression for the volume of this cross-section.

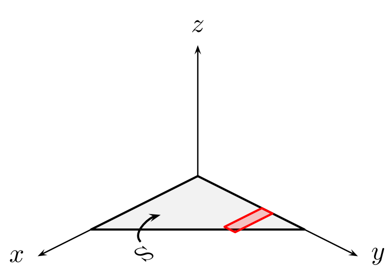

Figure 3: *Base S and rep. rect. in 3-D.*

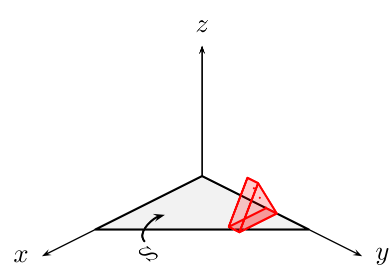

Figure 4: *Cross-section in 3-D.*

To find the volume of the cross-section, we need the area of the equilateral triangle (this is actually the base of the cross-sectional slice) and then thickness, which is Δy (this is actually the height of the cross-sectional slice.) In Figure 5, we've drawn the base of the cross-sectional slice. Since the cross-sections are equilateral triangles, each angle of the cross-section is $60°$. The right triangle formed by the altitude has base $\frac{-y+1}{2}$ because the altitude to the base of an isosceles triangle (and an equilateral triangle is certainly an isosceles triangle) bisects the base (and the vertex angle.) In a 30-60-90 triangle, the long leg (opposite the $60°$

[1] Stewart, *Calculus, Early Transcendentals*, p. 448, #56.

angle) is $\sqrt{3}$ times the short leg (opposite the 30° angle), so the altitude is $\sqrt{3} \cdot \frac{-y+1}{2}$. Thus, the area of the equilateral triangle is

$$\begin{aligned} A_{\text{triangle}} &= \frac{1}{2} \cdot (-y+1) \cdot \sqrt{3} \cdot \frac{-y+1}{2} \\ &= \frac{\sqrt{3}}{4} (-y+1)^2 \end{aligned}$$

and the volume of the cross-sectional slice is

$$\begin{aligned} V_{\text{cross-sectional slice}} &= \frac{\sqrt{3}}{4} (-y+1)^2 \, \Delta y \\ &= \frac{\sqrt{3}}{4} \left(y^2 - 2y + 1\right) \Delta y \end{aligned}$$

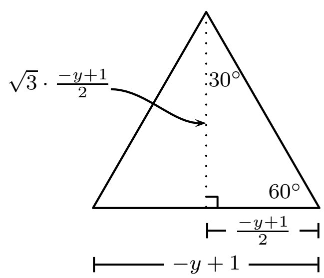

Figure 5: *Base of the cross-sectional solid.*

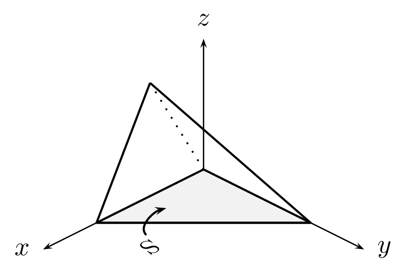

Figure 6: *The solid.*

We create cross-sectional slices from $y = 0$ to $y = 1$, so the volume of the solid is given by

$$\begin{aligned} V &= \int_{y=0}^{y=1} \frac{\sqrt{3}}{4} \left(y^2 - 2y + 1\right) \, \mathrm{d}y \\ &= \frac{\sqrt{3}}{4} \int_{y=0}^{y=1} \left(y^2 - 2y + 1\right) \, \mathrm{d}y \\ &= \frac{\sqrt{3}}{4} \left[\frac{y^3}{3} - y^2 + y\right]_{y=0}^{y=1} \\ &= \frac{\sqrt{3}}{4} \left[\left(\frac{1^3}{3} - 1^2 + 1\right) - \left(\frac{0^3}{3} - 0^2 + 0\right)\right] \\ &= \frac{\sqrt{3}}{4} \left[\frac{1}{3} - 1 + 1 - 0 + 0 - 0\right] \\ &= \frac{\sqrt{3}}{12} \end{aligned}$$

Thus, the volume of the solid, shown in Figure 6, whose base S is the triangular region with vertices (0,0), (1,0), and (0,1) and whose cross-sections perpendicular to the y-axis are equilateral triangles is $\frac{\sqrt{3}}{12}$ units^3.