Set up an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Then use your calculator to evaluate the integral correct to five decimal places.[1]

$$y = x^2, \quad x^2 + y^2 = 1, \quad y \geq 0$$

(a) About the x-axis.

The region bounded by the given curves is shown in Figure 1. We know $x^2+y^2 = 1$ is the unit circle, but because we are told $y \geq 0$, we only use the top half.

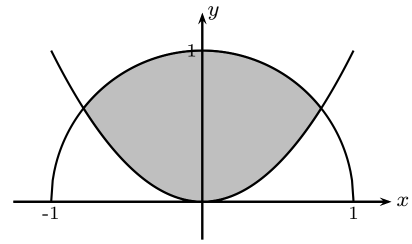


Figure 1

Before we can set up the integral for the volume of the solid of revolution, we need to find the coordinates of the points where the curves intersect. We solve

$$\begin{aligned} x^2 &= \sqrt{1-x^2} \\ x^4 &= 1 - x^2 \\ x^4 + x^2 - 1 &= 0 \end{aligned}$$

Let $w = x^2$, then $w^2 = x^4$ and we get

$$w^2 + w - 1 = 0$$

From the quadratic formula,

$$\begin{aligned} w &= \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot -1}}{2 \cdot 1} \\ w &= \frac{-1 \pm \sqrt{5}}{2} \end{aligned}$$

Since $w = x^2$, w must be a nonnegative number. Consider $w = \frac{-1-\sqrt{5}}{2}$. The numerator is negative and the denominator is positive, ot the value of w is negative. Thus

$$\begin{aligned} w &= \frac{-1 + \sqrt{5}}{2} \\ x^2 &= \frac{-1 + \sqrt{5}}{2} \end{aligned}$$

so

$$x = -\sqrt{\frac{-1+\sqrt{5}}{2}} \quad \text{or} \quad x = \sqrt{\frac{-1+\sqrt{5}}{2}}$$

In Figure 2, we've drawn in a representative rectangle of width Δx perpendicular to the axis of revolution, $y = 0$.

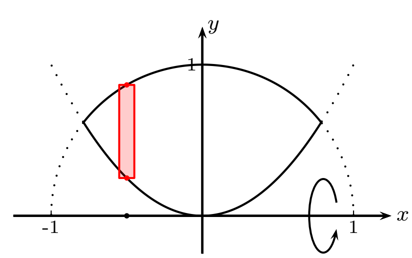


Figure 2

[1]Stewart, *Calculus, Early Transcendentals*, p. 447, #34.

In Figure 3, the washer corresponding to the representative rectangle is drawn to the left.

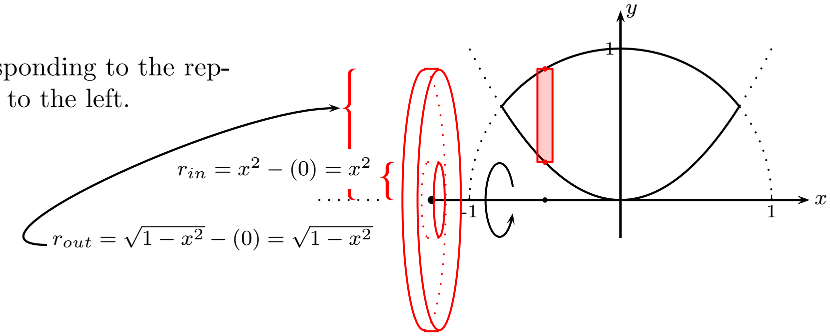


Figure 3

The volume V_w of the washer is given by

$$V_w = (\text{area of the base})\,\Delta x$$

where the base of the washer is the region formed by the two concentric circles.

$$\begin{aligned} V_w &= (\text{area of outside circle} - \text{area of inside circle})\,\Delta x \\ &= \left(\pi r_{out}^2 - \pi r_{in}^2\right)\Delta x \end{aligned}$$

Note that the outside radius r_{out} = *larger value* − *smaller value*, that is, $r_{out} = \sqrt{1-x^2} - (0) = \sqrt{1-x^2}$ and $r_{in} = x^2 - (0) = x^2$. So

$$V_w = \left(\pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(x^2\right)^2\right)\Delta x$$

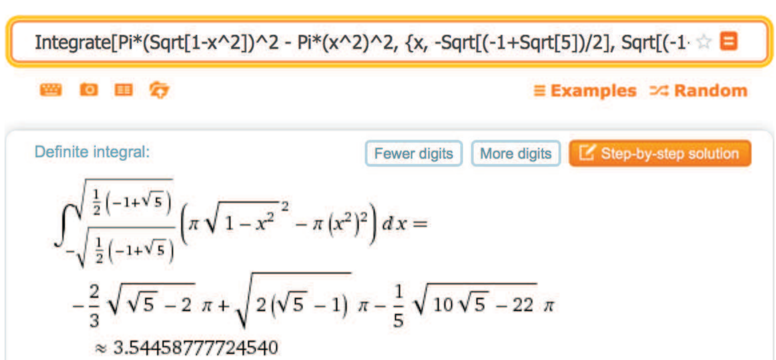
We create washers from $x = -\sqrt{\frac{-1+\sqrt{5}}{2}}$ to $x = \sqrt{\frac{-1+\sqrt{5}}{2}}$, so

$$\begin{aligned} \text{Volume}_{SOR} &= \lim_{n\to\infty}\sum_{i=1}^{n}\left(\pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(x^2\right)^2\right)\Delta x \\ &= \int_{x=-\sqrt{\frac{-1+\sqrt{5}}{2}}}^{x=\sqrt{\frac{-1+\sqrt{5}}{2}}} \pi\left(\sqrt{1-x^2}\right)^2 - \pi\left(x^2\right)^2\ \Delta x \end{aligned}$$

Using the input

```
Integrate[Pi*(Sqrt[1-x^2])^2 - Pi*(x^2)^2, {x, -Sqrt[(-1+Sqrt[5])/2], Sqrt[(-1+Sqrt[5])/2]}]
```

WolframAlpha gives us



Thus, the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $x^2 + y^2 = 1$ for $y \geq 0$ about the x-axis is ≈ 3.54459.

(b) About the y-axis.

When we revolve the region about the y-axis, there are three challenges we must contend with.

First, if we build the representative rectangle perpendicular to the axis of revolution (as we should) and from the left side to the right side, then the region will overlap itself and we'll get twice the volume we want. To correct this, we will build the representative rectangle from the y-axis to the right-hand side of the region. See Figure 4.

Second, the function on the right side of the region changes at the dotted line on Figure 4. This means we will need *two* distinct integrals; the volume will be the sum of those two integrals.

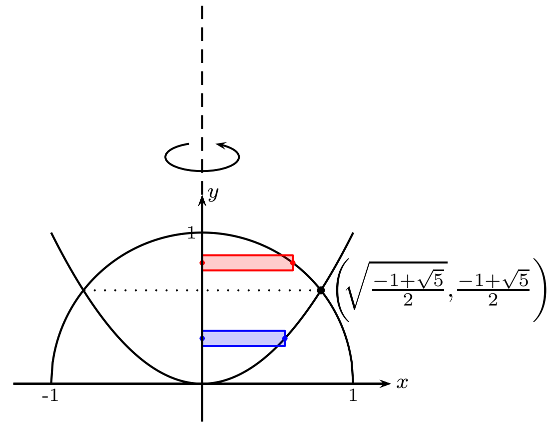


Figure 4

Finally, we must rewrite the given functions to express x as a function of y. We are given $y = x^2$, so $x = \pm\sqrt{y}$ and since the y values for the representative rectangles are all nonnegative, we take $x = \sqrt{y}$. Similarly, since $x^2 + y^2 = 1$, we take $x = \sqrt{1 - y^2}$.

For the red representative rectangle from $y = \left(-1 + \sqrt{5}\right)/2$ to $y = 1$, we have $r_{out} = \sqrt{1 - x^2}$ and $r_{in} = 0$. Thus

$$V_{red} = \int_{y=\left(-1+\sqrt{5}\right)/2}^{y=1} \pi \left(\sqrt{1 - y^2}\right)^2 - \pi \left(0\right)^2 \, dy$$

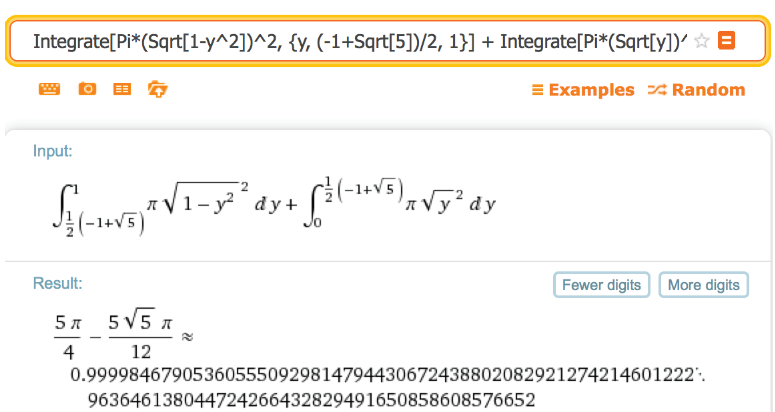
For the blue representative rectangle from $y = 0$ to $y = \left(-1 + \sqrt{5}\right)/2$, we have $r_{out} = \sqrt{y}$ and $r_{in} = 0$. Thus

$$V_{blue} = \int_{y=0}^{y=\left(-1+\sqrt{5}\right)/2} \pi \left(\sqrt{y}\right)^2 - \pi \left(0\right)^2 \, dy$$

Entering

```
Integrate[Pi*(Sqrt[1-y^2])^2, {y, (-1+Sqrt[5])/2, 1}] +
  Integrate[Pi*(Sqrt[y])^2, {y,0,(-1+Sqrt[5])/2}]
```

into WolframAlpha gives us



Thus, the volume of the solid obtained by revolving the region bounded by $y = x^2$ and $x^2 + y^2 = 1$ for $y \geq 0$ about the y-axis is ≈ 0.99998.