Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.[1]

$$y = x^3, \quad y = 1, \quad x = 2; \quad \text{about } y = -3$$

Let's sketch the graph so we have an idea of what we are working with.

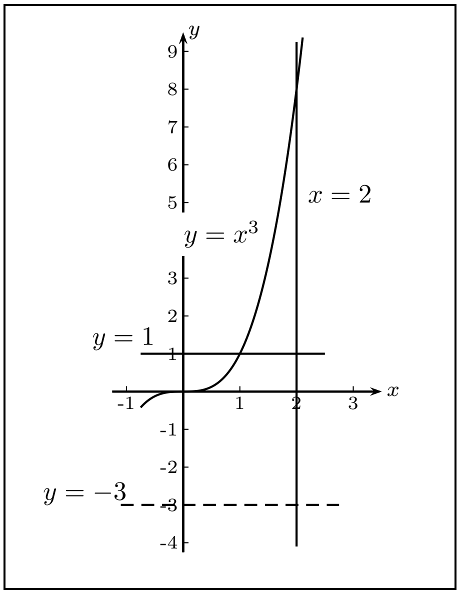

Figure 1

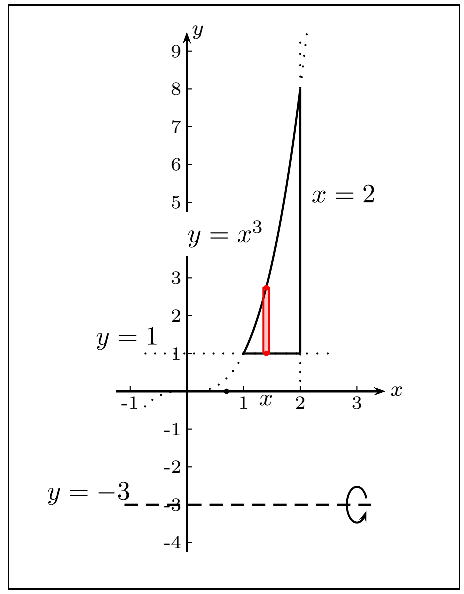

Figure 2

In Figure 1, we've graphed the functions $y = x^3$, $y = 1$, and the line $x = 2$ and can see the region bounded by those functions. The graphs intersect at $x = 1$ and $x = 2$, so those are the bounds on our region.

Figure 2 shows a representative rectangle of width Δx sketched in the region. We will rotate that representative rectangle about the line $y = -3$, determine the volume of the representative washer, and then add them all up from $x = 1$ to $x = 2$.

In Figure 3, the washer is drawn to the right. The volume V_w of the washer is given by

$$V_w = (\text{area of the base})\,\Delta x$$

where the base of the washer is the region formed by the two concentric circles.

$$\begin{aligned} V_w &= (\text{area of outside circle} - \text{area of inside circle})\,\Delta x \\ &= \left(\pi r_{out}^2 - \pi r_{in}^2\right)\Delta x \end{aligned}$$

Note that the outside radius r_{out} = *larger value* − *smaller value*, that is, $r_{out} = x^3 - (-3) = x^3 + 3$. Similarly for r_{in}. So

$$V_w = \left(\pi\left(x^3 + 3\right)^2 - \pi\,(4)^2\right)\Delta x$$

[1]Stewart, *Calculus, Early Transcendentals*, p. 446, #12.

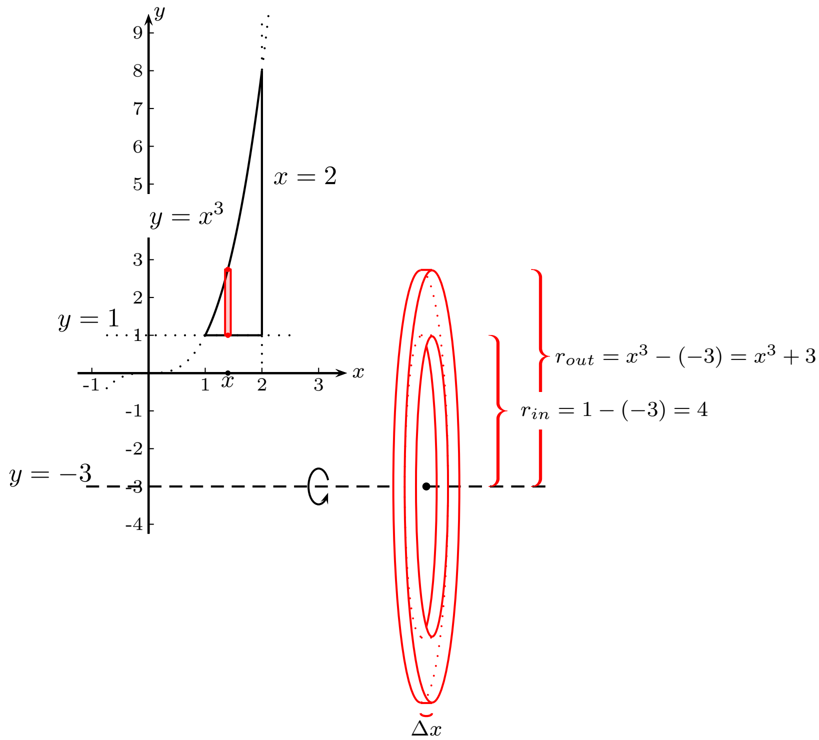

Figure 3

We create washers from $x = 1$ to $x = 2$, so

$$\begin{aligned}
\text{Volume}_{SOR} &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\pi\left(x^3+3\right)^2 - \pi\left(4\right)^2\right)\Delta x \\
&= \int_{x=1}^{x=2} \left(\pi\left(x^3+3\right)^2 - \pi\left(4\right)^2\right)\, dx \\
&= \int_{x=1}^{x=2} \pi\left(\left(x^3+3\right)^2 - \left(4\right)^2\right)\, dx \\
&= \pi \int_{x=1}^{x=2} x^6 + 6x^3 + 9 - 16\, dx \\
&= \pi \int_{x=1}^{x=2} x^6 + 6x^3 - 7\, dx
\end{aligned}$$

We find an antiderivative for the integrand, and apply the Fundamental Theorem of Calculus.

$$
\begin{aligned}
&= \pi \left[\frac{x^7}{7} + 6 \cdot \frac{x^4}{4} - 7x \right]_{x=1}^{x=2} \\
&= \pi \left[\left(\frac{(2)^7}{7} + \frac{3}{2} \cdot 2^4 - 7 \cdot 2 \right) - \left(\frac{(1)^7}{7} + \frac{3}{2} \cdot (1)^4 - 7 \cdot 1 \right) \right] \\
&= \pi \left[\left(\frac{128}{7} + \frac{3}{2} \cdot 16 - 14 \right) - \left(\frac{1}{7} + \frac{3}{2} \cdot 1 - 7 \right) \right] \\
&= \pi \left[\left(\frac{128}{7} + 10 \right) - \left(\frac{1}{7} + \frac{3}{2} - 7 \right) \right] \\
&= \pi \left[\frac{128}{7} + \frac{70}{7} - \frac{1}{7} - \frac{3}{2} + \frac{49}{7} \right] \\
&= \pi \left[\frac{256}{14} + \frac{140}{14} - \frac{2}{14} - \frac{21}{14} + \frac{98}{14} \right] \\
&= \pi \cdot \frac{471}{14} \\
&= \frac{471}{14} \pi
\end{aligned}
$$

Thus the volume of the solid obtained by rotating the region bounded by the graphs of $y = x^3$, $y = 1$, and $x = 2$ about the line $y = -3$ is $\frac{471}{14}\pi$ units^3.