Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.[1]

$$y = 6 - x^2, \quad y = 2; \quad \textit{about the } x\textit{-axis}$$

Let's sketch the graph so we have an idea of what we are working with.

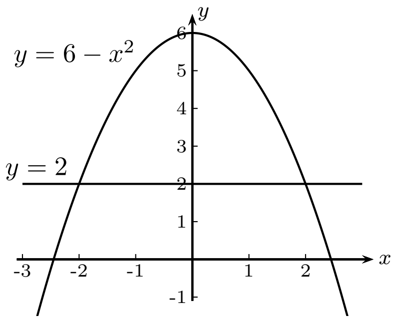

Figure 1

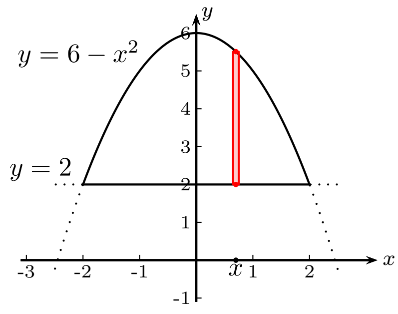

Figure 2

In Figure 1, we've graphed the functions $y = 6 - x^2$ and $y = 2$ and can see the region bounded by those functions. It appears that the graphs intersect at $x = \pm 2$; a little algebra should confirm that.

$$\begin{aligned} 6 - x^2 &= 2 \\ 0 &= x^2 - 4 \\ 0 &= (x+2)(x-2) \end{aligned}$$

and, by zero product property we get

$$x = -2 \quad \text{or} \quad x = 2$$

Figure 2 shows a representative rectangle of width Δx. We will rotate that representative rectangle about the x-axis, determine the volume of the washer, and then add them all up.

In Figure 3, the washer is drawn to the right. The volume V_w of the washer is given by

$$V_w = (\text{area of the base})\, \Delta x$$

where the base of the washer is the region formed by the two concentric circles.

$$\begin{aligned} &= (\text{area of outside circle} - \text{area of inside circle})\, \Delta x \\ &= \left(\pi r_{out}^2 - \pi r_{in}^2\right) \Delta x \\ &= \left(\pi \left(6 - x^2\right)^2 - \pi \left(2\right)^2\right) \Delta x \end{aligned}$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 446, #42.

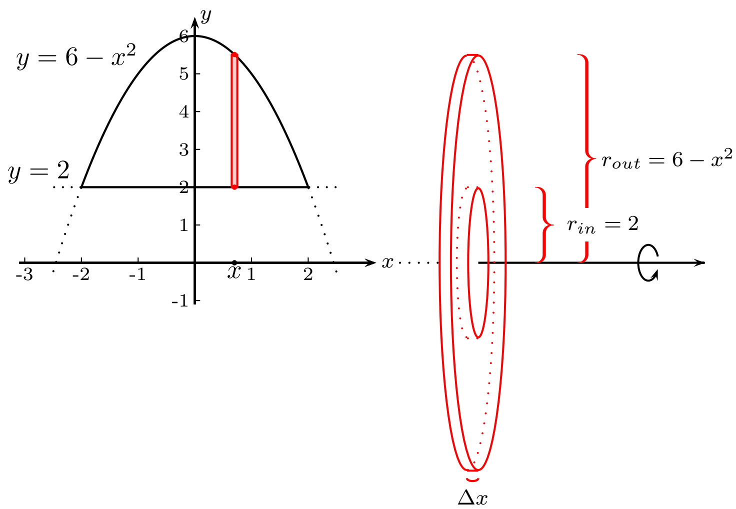

Figure 3

We create washers from $x = -2$ to $x = 2$, so

$$\begin{aligned}
\text{Volume}_{SOR} &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\pi \left(6 - x^2\right)^2 - \pi \left(2\right)^2\right) \Delta x \\
&= \int_{x=-2}^{x=2} \left(\pi \left(6 - x^2\right)^2 - \pi \left(2\right)^2\right) \, \mathrm{d}x \\
&= \int_{x=-2}^{x=2} \pi \left(\left(6 - x^2\right)^2 - \left(2\right)^2\right) \, \mathrm{d}x \\
&= \pi \int_{x=-2}^{x=2} 36 - 12x^2 + x^4 - 4 \, \mathrm{d}x \\
&= \pi \int_{x=-2}^{x=2} x^4 - 12x^2 + 32 \, \mathrm{d}x
\end{aligned}$$

We find an antiderivative for the integrand, and apply the Fundamental Theorem of Calculus.

$$\begin{aligned}
&= \pi \left[\frac{x^5}{5} - 12 \cdot \frac{x^3}{3} + 32x\right]_{x=-2}^{x=2} \\
&= \pi \left[\left(\frac{2^5}{5} - 4 \cdot 2^3 + 32 \cdot 2\right) - \left(\frac{\left(-2\right)^5}{5} - 4 \cdot \left(-2\right)^3 + 32 \cdot -2\right)\right] \\
&= \pi \left[\left(\frac{32}{5} - 4 \cdot 8 + 64\right) - \left(\frac{-32}{5} - 4 \cdot (-8) - 64\right)\right] \\
&= \pi \left[\left(\frac{32}{5} + 32\right) - \left(\frac{-32}{5} - 32\right)\right] \\
&= \pi \left[\frac{32}{5} + \frac{160}{5} + \frac{32}{5} + \frac{160}{5}\right] \\
&= \pi \cdot \frac{384}{5} \\
&= \frac{384}{5}\pi
\end{aligned}$$

Thus the volume of the solid obtained by rotating the region bounded by $y = 6 - x^2$ and $y = 2$ about the x-axis is $\frac{384}{5}\pi$ units^3.