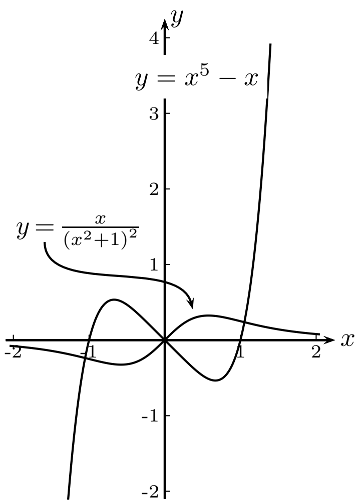
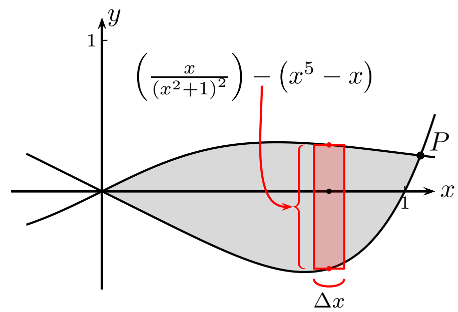
Use a graph to find approximate x-coordinates of the intersections of the given curves. Then find (approximately) the area of the region bounded by the curves.[1]

$$y = \frac{x}{\left(x^2+1\right)^2}, \quad y = x^5 - x, \quad x \geq 0$$

Let's sketch the graph so we have an idea of what we are working with.





The graph on the left shows the two functions, while the graph on the right shows the region of interest, along with a representative rectangle. Using `calc:intersect` on the TI-84, we find the x-coordinate of point P is $x \approx 1.0521$.

The base of the representative rectangle is Δx, and the height is $\left(\frac{x}{(x^2+1)^2}\right) - \left(x^5 - x\right)$, so the area of the representative rectangle is $\left(\left(\frac{x}{(x^2+1)^2}\right) - \left(x^5 - x\right)\right)\Delta x$. We generate representative rectangles from $x = 0$ to $x = 1.0521$, so the area is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} \left(\left(\frac{x}{\left(x^2+1\right)^2}\right) - \left(x^5 - x\right)\right)\Delta x$$
$$\approx \int_{x=0}^{x=1.0521} \left(\frac{x}{\left(x^2+1\right)^2}\right) - \left(x^5 - x\right)\, \mathrm{d}x$$

We'll split the integral

$$= \int_{x=0}^{x=1.0521} \frac{x}{\left(x^2+1\right)^2}\,\mathrm{d}x - \int_{x=0}^{x=1.0521} x^5 - x\,\mathrm{d}x$$

because we need to use substitution on the left integral. Let $u = x^2 + 1$, so $\mathrm{d}u = 2x\,\mathrm{d}x$. Let's multiply outside by $\frac{1}{2}$ and inside by 2

$$= \frac{1}{2}\int_{x=0}^{x=1.0521} \frac{2x}{\left(x^2+1\right)^2}\,\mathrm{d}x - \int_{x=0}^{x=1.0521} x^5 - x\,\mathrm{d}x$$

[1] Stewart, *Calculus, Early Transcendentals*, p. 435, #38.

Continued $\Longrightarrow$

When $x = 0$, $u = 1$, and when $x = 1.0521$, $u \approx 2.1069$, so we have

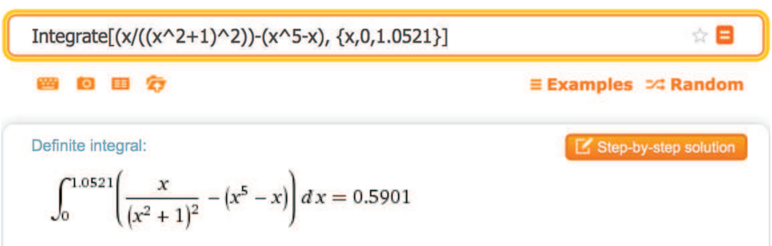
$$\begin{aligned}
&= \frac{1}{2}\int_{u=1}^{u=2.1069} \frac{1}{u^2}\,\mathrm{d}u - \int_{x=0}^{x=1.0521} x^5 - x\,\mathrm{d}x \\
&= \frac{1}{2}\int_{u=1}^{u=2.1069} u^{-2}\,\mathrm{d}u - \int_{x=0}^{x=1.0521} x^5 - x\,\mathrm{d}x \\
&= \frac{1}{2}\left[\frac{u^{-1}}{-1}\right]_{u=1}^{u=2.1069} - \left[\frac{x^6}{6} - \frac{x^2}{2}\right]_{x=0}^{x=1.0521} \\
&= \frac{1}{2}\left[-\frac{1}{u}\right]_{u=1}^{u=2.1069} - \left[\frac{x^6}{6} - \frac{x^2}{2}\right]_{x=0}^{x=1.0521} \\
&= \frac{1}{2}\left[\left(-\frac{1}{2.1069}\right) - \left(-\frac{1}{1}\right)\right] - \left[\left(\frac{1.0521^6}{6} - \frac{1.0521^2}{2}\right) - \left(\frac{0^6}{6} - \frac{0^2}{2}\right)\right]
\end{aligned}$$

After some careful calculator work, we get

$$\approx 0.5901$$

Thus, the area of the region bounded by $y = \frac{x}{(x^2+1)^2}$ and $y = x^5 - x$ for $x \geq 0$ is about 0.5901.

As we become more familiar with the concept and techniques of integration, we will learn a variety of numerical and technological approximation techniques. For example, WolframAlpha gives the following result for this problem:



Be sure you know what your teacher's expectations are **BEFORE** you use any technology to obtain your results.