For the given functions f and g, find (a) $f \circ g$, (b) $g \circ f$, (c) $f \circ f$, and (d) $g \circ g$. State the domain of each composite function.[1]

$$f(x) = \frac{2x-1}{x-2};\ g(x) = \frac{x+4}{2x-5}$$

a. $f \circ g$

Let's examine the individual functions first. For $f(x) = \frac{2x-1}{x-2}$, we must exclude $x = 2$ from the domain. For $g(x) = \frac{x+4}{2x-5}$, we must exclude $x = \frac{5}{2}$ from the domain. Since $g(x)$ is the input to f, we must also be sure that the output from $g(x)$ is never 2, because it becomes the input to $f(x)$. We solve

$$\begin{aligned}
g(x) &= 2 \\
\frac{x+4}{2x-5} &= 2 \\
\frac{x+4}{2x-5} \cdot (2x-5) &= 2 \cdot (2x-5), \quad 2x-5 \neq 0 \\
x+4 &= 4x-10 \\
-3x &= -14 \\
x &= \frac{14}{3}
\end{aligned}$$

So in addition to $x \neq \frac{5}{2}$, we also know $x \neq \frac{14}{3}$. Again, $x \neq \frac{5}{2}$ because the input function $g(x)$ would be undefined and $x \neq \frac{14}{3}$ because with that input to g, the function $f(x)$ would be undefined.

Thus the domain of $f(g(x))$ is all real numbers x such that $x \neq \frac{5}{2}$ or $x \neq \frac{14}{3}$. In set builder notation, we write $\left\{x | x \neq \frac{5}{2}, x \neq \frac{14}{3}\right\}$. Using interval notation, we write $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \frac{14}{3}\right) \cup \left(\frac{14}{3}, \infty\right)$.

Now that we know the domain, lets find the function rule for $f \circ g$.

$$\begin{aligned}
f \circ g &= f(g(x)) \\
&= f\left(\frac{x+4}{2x-5}\right) \\
&= \frac{2 \cdot \frac{x+4}{2x-5} - 1}{\frac{x+4}{2x-5} - 2} \\
&= \frac{\frac{2x+8}{2x-5} - 1}{\frac{x+4}{2x-5} - 2} \\
&= \frac{\frac{2x+8}{2x-5} - \frac{2x-5}{2x-5}}{\frac{x+4}{2x-5} - 2 \cdot \frac{2x-5}{2x-5}} \\
&= \frac{\frac{2x+8-2x+5}{2x-5}}{\frac{x+4}{2x-5} - \frac{4x-10}{2x-5}} \\
&= \frac{\frac{13}{2x-5}}{\frac{x+4-4x+10}{2x-5}} \\
&= \frac{\frac{13}{2x-5}}{\frac{-3x+14}{2x-5}} \\
&= \frac{13}{2x-5} \cdot \frac{2x-5}{-3x+14} \\
&= \frac{13}{-3x+14} \quad \text{or} \quad = -\frac{13}{3x-14}
\end{aligned}$$

[1] Sullivan, *Precalculus: Enhanced with Graphing Utilities*, p. 257, #44.

Thus, if $f(x) = \frac{2x-1}{x-2}$ and $g(x) = \frac{x+4}{2x-5}$, then

$$f \circ g = f(g(x)) = \frac{13}{-3x+14}$$

and the domain of $f \circ g$ is $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \frac{14}{3}\right) \cup \left(\frac{14}{3}, \infty\right)$.

b. $g \circ f$

$(g \circ f)(x) = g(f(x))$. Here, the input is $f(x) = \frac{2x-1}{x-2}$ and from part (**a**) we know that we must exclude $x = 2$ from its domain. For $g(x) = \frac{x+4}{2x-5}$, we must exclude $x = \frac{5}{2}$ from its domain. Since $f(x)$ is the input to g, we must also be sure that the output from $f(x)$ is never $\frac{5}{2}$, because it becomes the input to $g(x)$. We solve

$$\begin{aligned} f(x) &= \frac{5}{2} \\ \frac{2x-1}{x-2} &= \frac{5}{2} \\ \frac{2x-1}{x-2} \cdot 2(x-2) &= \frac{5}{2} \cdot 2(x-2), \quad x \neq 2 \\ 4x-2 &= 5x-10 \\ -x &= -8 \\ x &= 8 \end{aligned}$$

So in addition to $x \neq 2$, we also know $x \neq 8$. Again, $x \neq 2$ because the input function $f(x)$ would be undefined and $x \neq 8$ because with that input to f, the function $g(x)$ would be undefined.

Thus the domain of $g(f(x))$ is all real numbers x such that $x \neq 2$ and $x \neq 8$. In set builder notation, we write $\{x | x \neq 2, x \neq 8\}$. Using interval notation, we write $(-\infty, 2) \cup (2, 8) \cup (8, \infty)$.

Now that we know the domain, lets find the function rule for $g \circ f$.

$$\begin{aligned} g \circ f &= g(f(x)) \\ &= g\left(\frac{2x-1}{x-2}\right) \\ &= \frac{\frac{2x-1}{x-2} + 4}{2 \cdot \frac{2x-1}{x-2} - 5} \\ &= \frac{\frac{2x-1+4(x-2)}{x-2}}{\frac{4x-2-5(x-2)}{x-2}} \\ &= \frac{\frac{6x-9}{x-2}}{\frac{-x+8}{x-2}} \\ &= \frac{6x-9}{x-2} \cdot \frac{x-2}{-x+8} \\ &= \frac{6x-9}{-x+8} \quad \text{or} \quad = -\frac{3(2x-3)}{x-8} \end{aligned}$$

Thus, if $f(x) = \frac{2x-1}{x-2}$ and $g(x) = \frac{x+4}{2x-5}$, then

$$g \circ f = g(f(x)) = \frac{6x-9}{-x+8}$$

and the domain of $g \circ f$ is $(-\infty, 2) \cup (2, 8) \cup (8, \infty)$.

c. $f \circ f$

$(f \circ f)(x) = f(f(x))$. Here, the input is $f(x) = \frac{2x-1}{x-2}$ and from part (**a**) we know that we must exclude $x = 2$ from the domain. Since $f(x)$ is also the input to f, we must be sure that the output from $f(x)$ is never 2, because it becomes the input to $f(x)$. We solve

$$\begin{aligned} f(x) &= 2 \\ \frac{2x-1}{x-2} &= 2 \\ \frac{2x-1}{x-2} \cdot (x-2) &= 2 \cdot (x-2), \quad x \neq 2 \\ 2x - 1 &= 2x - 4 \\ -1 &= -4 \end{aligned}$$

This last equation is never true, and that means there are *no* values of x for which $f(x) = 2$.

Thus the domain of $f(f(x))$ is all real numbers x such that $x \neq 2$. In set builder notation, we write $\{x|x \neq 2\}$. Using interval notation, we write $(-\infty,2) \cup (2, \infty)$.

Now that we know the domain, lets find the function rule for $f \circ f$.

$$\begin{aligned} f \circ f &= f(f(x)) \\ &= f\left(\frac{2x-1}{x-2}\right) \\ &= \frac{2 \cdot \frac{2x-1}{x-2} - 1}{\frac{2x-1}{x-2} - 2} \\ &= \frac{\frac{4x-2-1\cdot(x-2)}{x-2}}{\frac{2x-1-2(x-2)}{x-2}} \\ &= \frac{\frac{3x}{x-2}}{\frac{3}{x-2}} \\ &= \frac{3x}{x-2} \cdot \frac{x-2}{3} \\ &= x \end{aligned}$$

Thus, if $f(x) = \frac{2x-1}{x-2}$, then

$$f \circ f = f(f(x)) = x$$

and the domain of $f \circ f$ is $(-\infty,2) \cup (2, \infty)$.

d. $g \circ g$

$g \circ g(x) = g(g(x))$. Here, the input is $g(x) = \frac{x+4}{2x-5}$ and from part (**a**) we know that we must exclude $x = \frac{5}{2}$ from the domain. Since $g(x)$ is also the input to g, we must be sure that the output from $g(x)$ is never $\frac{5}{2}$, because it becomes the input to $g(x)$. We solve

$$\begin{aligned} g(x) &= \frac{5}{2} \\ \frac{x+4}{2x-5} &= \frac{5}{2} \\ \frac{x+4}{2x-5} \cdot 2 \cdot (2x-5) &= \frac{5}{2} \cdot 2 \cdot (2x-5), \quad x \neq \frac{5}{2} \\ 2(x+4) &= 5(2x-5) \\ 2x + 8 &= 10x - 25 \\ -8x &= -33 \\ x &= \frac{33}{8} \end{aligned}$$

So in addition to $x \neq \frac{5}{2}$, we also know $x \neq \frac{33}{8}$. Again, $x \neq \frac{5}{2}$ because the input function $g(x)$ would be undefined and $x \neq \frac{33}{8}$ because with that input to g, the function $f(x)$ would be undefined.

Thus the domain of $g(g(x))$ is all real numbers x such that $x \neq \frac{5}{2}$ and $x \neq \frac{33}{8}$. In set builder notation, we write $\left\{x | x \neq \frac{5}{2}, x \neq \frac{33}{8}\right\}$. Using interval notation, we write $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \frac{33}{8}\right) \cup \left(\frac{33}{8}, \infty\right)$.

Now that we know the domain, lets find the function rule for $g \circ g$.

$$\begin{aligned} g \circ g &= g(g(x)) \\ &= g\left(\frac{x+4}{2x-5}\right) \\ &= \frac{\frac{x+4}{2x-5} + 4}{2 \cdot \frac{x+4}{2x-5} - 5} \\ &= \frac{\frac{x+4+4(2x-5)}{2x-5}}{\frac{2x+8-5(2x-5)}{2x-5}} \\ &= \frac{\frac{9x-16}{2x-5}}{\frac{-8x+33}{2x-5}} \\ &= \frac{9x-16}{2x-5} \cdot \frac{2x-5}{-8x+33} \\ &= \frac{9x-16}{-8x+33} \quad \text{or} \quad = -\frac{9x-16}{8x-33} \end{aligned}$$

Thus, if $g(x) = \frac{x+4}{2x-5}$, then

$$g \circ g = g(g(x)) = \frac{9x-16}{-8x+33}$$

and the domain of $g \circ g$ is $\left(-\infty, \frac{5}{2}\right) \cup \left(\frac{5}{2}, \frac{33}{8}\right) \cup \left(\frac{33}{8}, \infty\right)$.