

Precalculus, Section 4.3, #28
Complex Zeros; Fundamental Theorem of Algebra

Use the given zero to find the remaining zeros of each function¹

$$f(x) = x^4 - 7x^3 + 14x^2 - 38x - 60; \quad \text{zero: } 1 + 3i$$

We are given the zero $1 + 3i$. From the Conjugate Pairs Theorem, we know that $1 - 3i$ must also be a zero. From the Fundamental Theorem of Algebra, we know that our function has factors

$$(x - (1 + 3i)) \quad \text{and} \quad (x - (1 - 3i))$$

If we multiply these two factors together, the product will also be a factor of $f(x)$. (This is the same as saying 2 and 3 are both factors of 12, so $2 \cdot 3 = 6$ is also a factor of 12.)

$$\begin{aligned}(x - (1 + 3i))(x - (1 - 3i)) &= (x - 1 - 3i)(x - 1 + 3i) \\ &= x^2 - x + 3xi - x + 1 - 3i - 3xi + 3i - (3i)^2 \\ &= x^2 - 2x + 1 - (-9) \\ &= x^2 - 2x + 10\end{aligned}$$

Now we will divide our function by this factor to find the other factor. (This is the same as saying we know 6 is a factor of 12, so we can divide $12 \div 6 = 2$ to find the other factor of 2.)

The image shows a handwritten polynomial long division on a grid background. The divisor is $x^2 - 2x + 10$ and the dividend is $x^4 - 7x^3 + 14x^2 - 38x - 60$. The quotient is $x^2 - 5x - 6$. The steps are as follows:
1. $x^2 - 2x + 10$ goes into $x^4 - 7x^3 + 14x^2 - 38x - 60$ x^2 times.
2. Subtract $(x^4 - 2x^3 + 10x^2)$ from the dividend to get $-5x^3 + 4x^2 - 38x - 60$.
3. $x^2 - 2x + 10$ goes into $-5x^3 + 4x^2 - 38x - 60$ $-5x$ times.
4. Subtract $(-5x^3 + 10x^2 - 50x)$ from the remainder to get $-6x^2 + 12x - 60$.
5. $x^2 - 2x + 10$ goes into $-6x^2 + 12x - 60$ -6 times.
6. Subtract $(-6x^2 + 12x - 60)$ from the remainder to get 0.

This division tells us

$$x^4 - 7x^3 + 14x^2 - 38x - 60 = (x^2 - 5x - 6)(x^2 - 2x + 10)$$

and we factor to get

$$= (x - 6)(x + 1)(x^2 - 2x + 10)$$

To find the zeros, we solve $f(x) = 0$, so

$$\begin{aligned}x^4 - 7x^3 + 14x^2 - 38x - 60 &= 0 \\ (x - 6)(x + 1)(x^2 - 2x + 10) &= 0\end{aligned}$$

and from Zero Product Property, we have

$$x = 6, \quad x = -1, \quad x = 1 + 3i, \quad \text{and} \quad x = 1 - 3i$$

Thus the zeros of $f(x) = x^4 - 7x^3 + 14x^2 - 38x - 60$ are 6, -1 , $1 + 3i$, and $1 - 3i$.

¹Sullivan, *Precalculus: Enhanced with Graphing Utilities*, p. 215, #28.