Solve The system does not have a unique solution. Solve the system if possible. ¹

\[
\begin{align*}
  x + 5y - 2z &= 5 \\
  x + 3y - 2z &= 23 \\
  x + 4y - 2z &= 14
\end{align*}
\]

Begin by labeling the equations so we can keep track of what we’re doing.

\[
\begin{align*}
  x + 5y - 2z &= 5 \quad \text{(A)} \\
  x + 3y - 2z &= 23 \quad \text{(B)} \\
  x + 4y - 2z &= 14 \quad \text{(C)}
\end{align*}
\]

The \( x \) in equation (A) already has a coefficient of 1 so we can use it to eliminate the \( x \)'s in equations (B) and (C) as follows:

\[
\begin{align*}
  \text{(B)} - \text{(A)} &\rightarrow \text{ (B)} \quad \text{and} \quad \text{(C)} - \text{(A)} \rightarrow \text{ (C)} \\
  x + 5y - 2z &= 5 \quad \text{(A)} \\
  -2y &= 18 \quad \text{(B)} \\
  -1y &= 9 \quad \text{(C)}
\end{align*}
\]

Multiply (B) by \(-\frac{1}{2}\) to get 1 as the coefficient on the \( y \).

\[
\begin{align*}
  -\frac{1}{2} \text{ (B)} &\rightarrow \text{ (B)} \\
  x + 5y - 2z &= 5 \quad \text{(A)} \\
  y &= -9 \quad \text{(B)} \\
  -1y &= 9 \quad \text{(C)}
\end{align*}
\]

Add equations (B) and (C) to eliminate \( y \) in equation (C).

\[
\begin{align*}
  \text{(C)} + \text{(B)} &\rightarrow \text{ (C)} \\
  x + 5y - 2z &= 5 \quad \text{(A)} \\
  y &= -9 \quad \text{(B)} \\
  0 &= 0 \quad \text{(C)}
\end{align*}
\]

Notice how equation (C) is the true statement, \( 0 = 0 \). This means that the system has infinitely many solutions.

Since we don’t have a solution for \( z \) we say that \( z \) can be any number and back-substitute from there.

From equation (B) we know that \( y = -9 \) and we can use this information to solve for \( x \) in terms of \( z \) using equation (A).

\[
\begin{align*}
  x + 5y - 2z &= 5 \\
  x + 5(-9) - 2z &= 5 \\
  x &= 5 + 45 + 2z \\
  x &= 50 + 2z \\
  x &= 2z + 50
\end{align*}
\]

There are infinitely many solutions of the form \((2z + 50, -9, z)\).

¹Harshbarger/Yocco, College Algebra In Context, 5e, p. 519 #22.
Another way we can achieve the same result is by using matrices. Begin with the augmented matrix...

\[
\begin{bmatrix}
1 & 5 & -2 & 5 \\
1 & 3 & -2 & 23 \\
1 & 4 & -2 & 14 \\
\end{bmatrix}
\]

and use reduced row-echelon form (rref) on your calculator to get the solution matrix...

\[
\begin{bmatrix}
1 & 5 & -2 & 5 \\
0 & 1 & 0 & -9 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The bottom row of all zeros tells us that the system has infinitely many solutions and we can let \( z \) equal any number.

The middle row says \( y = -9 \) and when we substitute and solve the top row for \( x \) we get \( x = 2z + 50 \).

So there are infinitely many solutions of the form \((2z + 50, -9, z)\).